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Adrienne Wooten

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CHAPTER 1 Right Triangle Relationships and Trigonometry

Chapter Outline

1.1 RIGHT TRIANGLES AND SIMILARITY

1.2 PYTHAGOREAN THEOREM AND ITS CONVERSE

1.3 APPLICATIONS USING THE PYTHAGOREAN THEOREM

1.4 SPECIAL RIGHT TRIANGLES

1.5 SINE, COSINE, TANGENT

1.6 TRIGONOMETRIC RATIOS WITH A CALCULATOR

1.7 INVERSE TRIGONOMETRIC RATIOS

1.8 APPLICATION PROBLEMS

1.1 Right Triangles and Similarity

Here you'll learn the Inscribed Similar Triangles Theorem that states that a line drawn from the right angle of a right triangle perpendicular to the opposite side of the triangle creates similar triangles. You'll then use this fact to find missing lengths in right triangles.

Are all right triangles similar? Are there any short cuts to find missing measures when you have two similar right triangles? We will build on our knowledge of Inscribed Similar Triangles from the previous chapter. For a quick review, watch the video below.

Watch This

MEDIA

Click image to the left for more content.

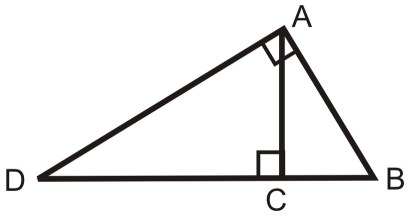
CK-12 Foundation: Inscribed Similar Triangles

Guidance

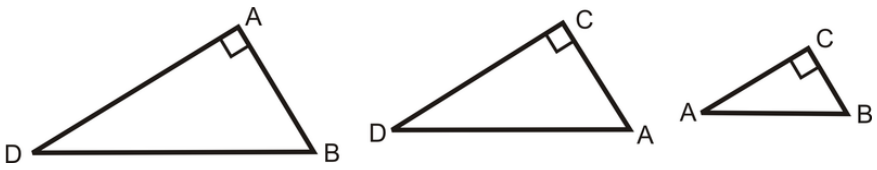
Remember that if two objects are similar, their corresponding angles are congruent and their sides are proportional in length. The altitude of a right triangle creates similar triangles. Recall the following theorem from the previous chapter.

Inscribed Similar Triangles Theorem: If an altitude is drawn from the right angle of any right triangle, then the two triangles formed are similar to the original triangle and all three triangles are similar to each other.

In $\triangle ADB, m\angle A = 90^\circ$ and $\overline{AC} \perp \overline{DB}$:



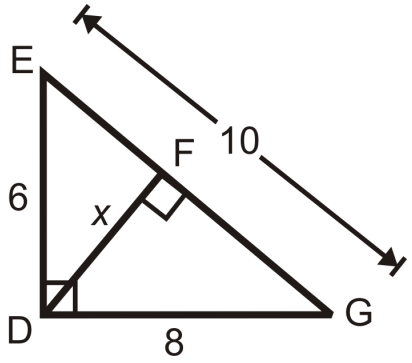
So, $\triangle ADB \sim \triangle CDA \sim \triangle CAB$:



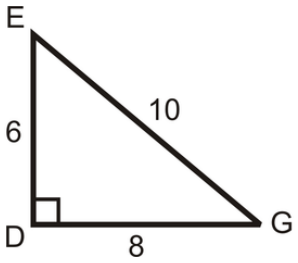
This means that all of the corresponding sides are proportional. You can use this fact to find missing lengths in right triangles.

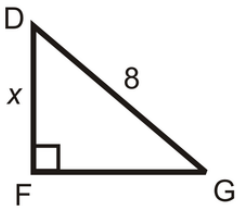
Example A

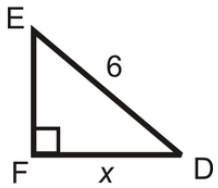
Find the value of x.



Separate the triangles to find the corresponding sides.





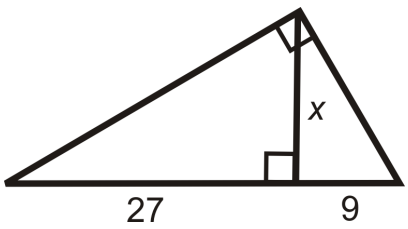


Set up a proportion.

$$\frac{\text{shorter leg in } \triangle EDG}{\text{shorter leg in } \triangle DFG} = \frac{\text{hypotenuse in } \triangle EDG}{\text{hypotenuse in } \triangle DFG}$$
$$\frac{6}{x} = \frac{10}{8}$$
$$48 = 10x$$
$$4.8 = x$$

Example B

Find the value of x.



Set up a proportion.

$$\frac{\text{shorter leg of smallest } \triangle}{\text{shorter leg of middle } \triangle} = \frac{\text{longer leg of smallest } \triangle}{\text{longer leg of middle } \triangle}$$
$$\frac{9}{x} = \frac{x}{27}$$
$$x^2 = 243$$
$$x = \sqrt{243} = 9\sqrt{3}$$

Geometric Mean

The geometric mean of any positive numbers a and b is the positive number x such that $\frac{a}{x} = \frac{x}{b}$.

For example, what is the geometric mean of 8 and 2?

$\frac{8}{x} = \frac{x}{2}$ Definition of geometric mean

$x^2 = 16$ Cross Products Property

$x = 4$ Per the definition of a geometric mean, we take the positive root

So 4 is the geometric mean of 8 and 2.

What does this have to do with similar right triangles? The altitude of inscribed right triangle is a geometric mean. If you refer back to Example B, x is also the geometric mean. We can find x with the following proportion:

$\frac{27}{x} = \frac{x}{9}$ Definition of geometric mean (Note that $\frac{9}{x} = \frac{x}{27}$ would yield the same answer)

$x^2 = 243$ Cross Products Property

$x = \sqrt{243}$

For additional information, click here.

The relationship between the two sides that was just discussed is a corollary to the Inscribed Similar Triangles Theorem (see Corollary 1 below). The geometric mean can also be used to find a leg or hypotenuse. This is represented by Corollary 2 below. Consider the diagram below as you look at the corollaries.

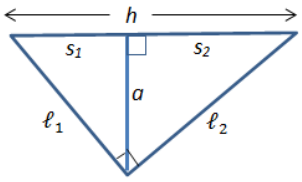


FIGURE 1.1

Corollary 1

$\frac{s_1}{a} = \frac{a}{s_2}$

Corollary 2

$\frac{h}{l_1} = \frac{l_1}{s_1}, \frac{h}{l_2} = \frac{l_2}{s_2}$

For an example using both corollaries, see Example C below.

Example C

What are the values of x and y in Figure 1.2?

Use Corollary 2:

$\frac{6+10}{x} = \frac{x}{6}$ Write a proportion.

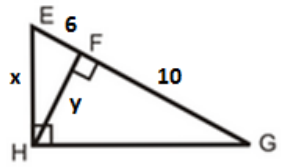


FIGURE 1.2

$x^2 = 96$ Cross Products Property

$x = \sqrt{96}$ Take the positive root.

Use Corollary 1:

$\frac{6}{y} = \frac{y}{10}$ Write a proportion.

$y^2 = 60$ Cross Products Property

$y = \sqrt{60}$ Take the positive root.

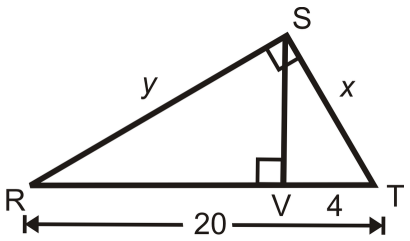
Now you have multiple strategies to solve for missing sides of inscribed right triangles: setting up proportions with corresponding sides or using the corollaries.

So are all right triangles similar?

Click here to watch a short video that will help you answer the question. In addition, it will give you a sneak preview of a few trigonometric ratios which will discuss later in the chapter.

Guided Practice

1. Find the value of x.



2. Now find the value of y in $\triangle RST$ above.

Answers:

1. Set up a proportion using corollary 2:

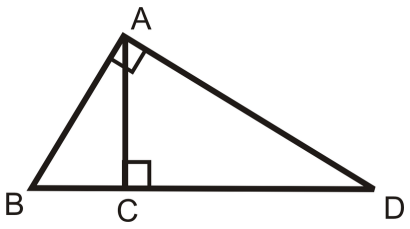
$$\frac{20}{x} = \frac{x}{4}$$
$$x^2 = 80$$
$$x = \sqrt{80}$$

2. Use the Pythagorean Theorem.

$$y^2 + \left(4\sqrt{5}\right)^2 = 20^2$$
$$y^2 + 80 = 400$$
$$y^2 = 320$$
$$y = \sqrt{320} = 8\sqrt{5}$$

Practice

Fill in the blanks.



1. Which segment in the figure is the altitude?
2. The altitude of an inscribed right triangle is also a _____ _____.
3. Using the geometric mean, fill in the following: $\frac{BC}{?} = \frac{?}{CD}$
4. Using Corollary 2, fill in the following: $\frac{?}{AB} = \frac{AB}{?}$
5. Solve for x and y (Figure 1.3).

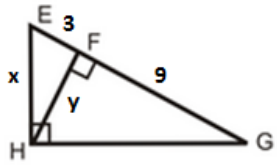


FIGURE 1.3

6. Solve for x and y (Figure 1.4).

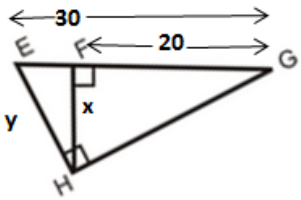


FIGURE 1.4

7. Find the value of x (Figure 1.5).

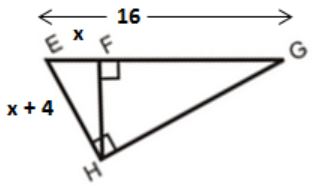


FIGURE 1.5

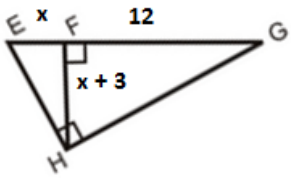


FIGURE 1.6

8. Find the value of x (Figure 1.6).

9. $\overline{CD}$ is the altitude to the hypotenuse of right ΔABC. The coordinates of A, D, and B are (3, 2), (3, 6), and (3, 15), respectively. Find all possible coordinates of C.

10. Are all right triangles similar? Explain.

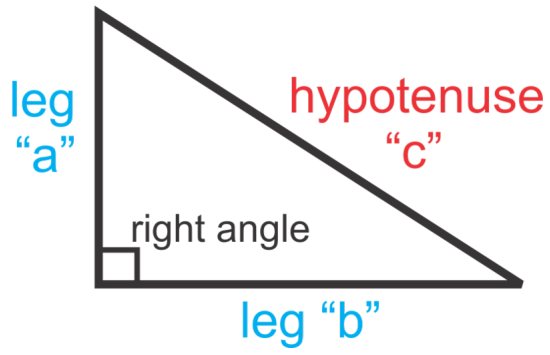
1.2 Pythagorean Theorem and its Converse

Here you'll discover, prove and apply the Pythagorean Theorem to solve for unknown sides in right triangles and prove triangles are right triangles.

Mr. Aubel wants to rope off half of his rectangular garden plot to keep the deer out. He will run the rope around the outside of the garden and diagonally down the center to form a right triangle. The garden measures 5 yards by 8 yards. How many full yards of rope does Mr. Aubel need?

Guidance

The Pythagorean Theorem refers to the relationship between the lengths of the three sides in a right triangle. It states that if a and b are the legs of the right triangle and c is the hypotenuse, then $a^2 + b^2 = c^2$. For example, the lengths 3, 4, and 5 are the sides of a right triangle because $3^2 + 4^2 = 5^2 (9 + 16 = 25)$. Keep in mind that c is always the longest side.

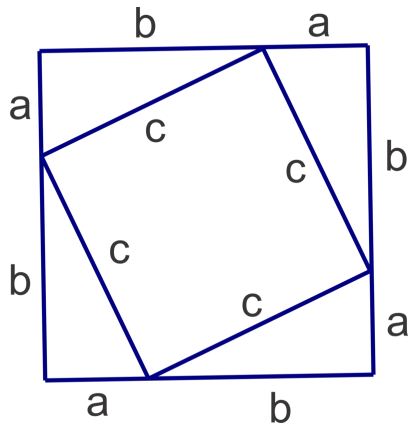


The converse of this statement is also true. If, in a triangle, c is the length of the longest side and the shorter sides have lengths a and b, and $a^2 + b^2 = c^2$, then the triangle is a right triangle.

Proof of Pythagorean Theorem

There are many proofs of the Pythagorean Theorem and here is one of them. We will be using the concept that the area of a figure is equal to the sum of the areas of the smaller figures contained within it and algebra to derive the Pythagorean Theorem.

Using the figure below (a square with a smaller square inside), first write two equations for its area, one using the lengths of the sides of the outer square and one using the sum of the areas of the smaller square and the four triangles.



Area 1: $(a+b)^2 = a^2 + 2ab + b^2$

Area 2: $c^2 + 4\left(\frac{1}{2}ab\right) = c^2 + 2ab$

Now, equate the two areas and simplify:

$$a^2 + 2ab + b^2 = c^2 + 2ab$$
$$a^2 + b^2 = c^2$$

Example A

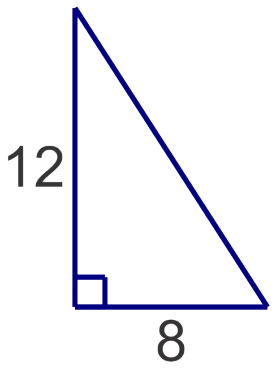
In a right triangle $a = 7$ and $c = 25$, find the length of the third side.

Solution: We can start by substituting what we know into the Pythagorean Theorem and then solve for the unknown side, b:

$$7^2 + b^2 = 25^2$$
$$49 + b^2 = 625$$
$$b^2 = 576$$
$$b = 24$$

Example B

Find the length of the third side of the triangle below. Leave your answer in reduced radical form.



Solution: Since we are given the lengths of the two legs, we can plug them into the Pythagorean Theorem and find the length of the hypotenuse.

$$\begin{aligned} 8^2 + 12^2 &= c^2 \\ 64 + 144 &= c^2 \\ c^2 &= 208 \\ c &= \sqrt{208} = \sqrt{16 \cdot 13} = 4\sqrt{13} \end{aligned}$$

Example C

Determine whether a triangle with lengths 21, 28, 35 is a right triangle.

Solution: We need to see if these values will satisfy $a^2 + b^2 = c^2$. If they do, then a right triangle is formed. So,

$$\begin{aligned} 21^2 + 28^2 &= 441 + 784 = 1225 \\ 35^2 &= 1225 \end{aligned}$$

Yes, the Pythagorean Theorem is satisfied by these lengths and a right triangle is formed by the lengths 21, 28 and 35.

Concept Problem Revisit

We are looking for the perimeter of the triangle. We are given the lengths of the sides so we need to find the hypotenuse.

Let's use the Pythagorean Theorem.

$$\begin{aligned} 5^2 + 8^2 &= c^2 \\ 25 + 64 &= c^2 \\ 89 &= c^2 \\ c &= \sqrt{89} \end{aligned}$$

Now to find the perimeter of the triangle, add the lengths of the three sides.

$5 + 8 + \sqrt{89} = 22.43$

Therefore, Mr. Aubel will need 23 yards of rope.

Guided Practice

For the given two sides, determine the length of the third side if the triangle is a right triangle.

1. $a = 10$ and $b = 5$

2. $a = 5$ and $c = 13$

Use the Pythagorean Theorem to determine if a right triangle is formed by the given lengths.

3. 16, 30, 34

4. 9, 40, 42

5. 2, 2, 4

Answers

1. $\sqrt{10^2+5^2} = \sqrt{100+25} = \sqrt{125} = 5\sqrt{5}$
2. $\sqrt{13^2-5^2} = \sqrt{169-25} = \sqrt{144} = 12$
3. $$16^2+30^2 = 256+900 = 1156$$ $$34^2 = 1156$$

Yes, this is a right triangle.

4. $$9^2+40^2 = 81+1600 = 1681$$ $$42^2 = 1764$$

No, this is not a right triangle.

5. This one is tricky, in a triangle the lengths of any two sides must have a sum *greater* than the length of the third side. These lengths do not meet that requirement so not only do they not form a *right* triangle, they do not make a triangle at all.

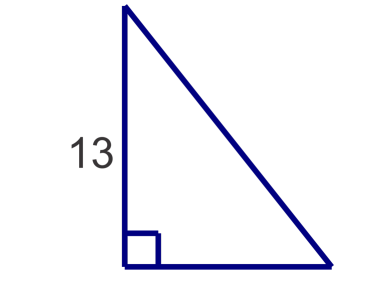
Vocabulary

The Pythagorean Theorem
If a and b are the legs of the right triangle and c is the hypotenuse, then $a^2+b^2=c^2$.

Converse of the Pythagorean Theorem
If c is the length of the longest side in a triangle and the shorter sides have lengths a and b, and $a^2+b^2=c^2$, then the triangle is a right triangle.

Pratice

Find the unknown side length for each right triangle below.



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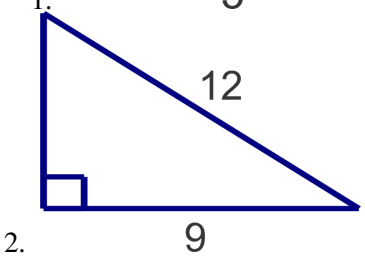

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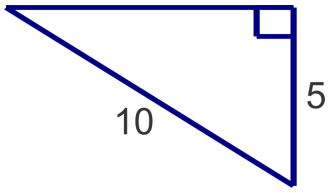
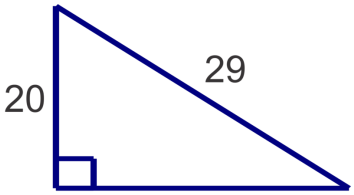


3.

4. $a = 6, b = 8$
5. $b = 6, c = 14$
6. $a = 12, c = 18$

Determine whether the following triangles are right triangles.



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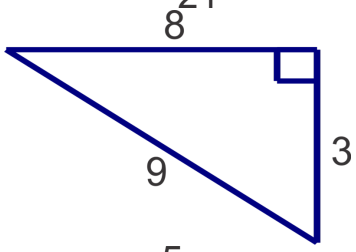

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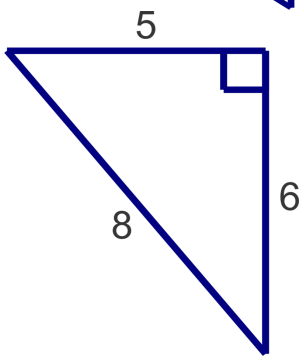
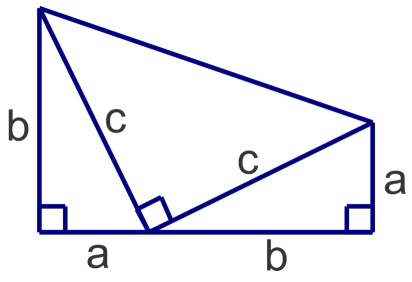


9.

Do the lengths below form a right triangle? Remember to make sure that they form a triangle.

10. 3, 4, 5
11. 6, 6, 11
12. 11, 13, 17

Major General James A. Garfield (and former President of the U.S.) is credited with deriving this proof of the Pythagorean Theorem using a trapezoid. Follow the steps to recreate his proof.



13. Find the area of the trapezoid using the trapezoid area formula: $A = \frac{1}{2}(b_1 + b_2)h$
14. Find the sum of the areas of the three right triangles in the diagram.
15. The areas found in the previous two problems should be the same value. Set the expressions equal to each other and simplify to get the Pythagorean Theorem.

1.3 Applications Using the Pythagorean Theorem

Here you'll learn how to apply the Pythagorean Theorem and its converse to solve real-world problems.

What if a fireman needed to rescue a cat from a tree? The cat is 40 feet up from the ground. The fireman places a ladder 30 feet away from the base of the tree? How tall does the ladder need to be for him to reach the cat? After completing this Concept, you'll be able to solve real-world applications like this one using the Pythagorean Theorem and its converse.

Guidance

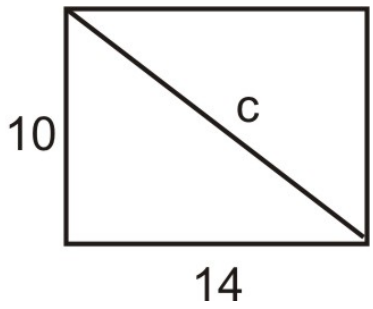
The Pythagorean Theorem and its converse have many applications for finding lengths and distances.

Example A

Maria has a rectangular cookie sheet that measures 10 *inches* $\times$ 14 *inches. Find the length of the diagonal of the cookie sheet.*

Solution

Draw a sketch:



Define variables: Let $c =$ length of the diagonal.

Write a formula: Use the Pythagorean Theorem: $a^2 + b^2 = c^2$

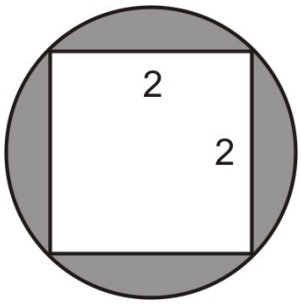
Solve the equation:

$$10^2 + 14^2 = c^2$$
$$100 + 196 = c^2$$
$$c^2 = 296 \Rightarrow c = \sqrt{296} \Rightarrow c = 2\sqrt{74} \text{ or } c = 17.2 \textit{ inches}$$

Check: $10^2 + 14^2 = 100 + 196 = 296$ and $c^2 = 17.2^2 = 296$. The solution checks out.

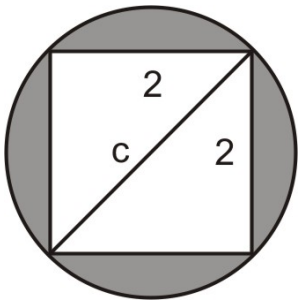
Example B

Find the area of the shaded region in the following diagram:



Solution

Draw the diagonal of the square in the figure:



Notice that the diagonal of the square is also the diameter of the circle.

Define variables: Let $c =$ diameter of the circle.

Write the formula: Use the Pythagorean Theorem: $a^2 + b^2 = c^2$.

Solve the equation:

$$\begin{aligned} 2^2 + 2^2 &= c^2 \\ 4 + 4 &= c^2 \\ c^2 = 8 \Rightarrow c &= \sqrt{8} \Rightarrow c = 2\sqrt{2} \end{aligned}$$

The diameter of the circle is $2\sqrt{2}$, therefore the radius $R = \sqrt{2}$.

Area of a circle formula: $A = \pi \cdot R^2 = \pi\left(\sqrt{2}\right)^2 = 2\pi$.

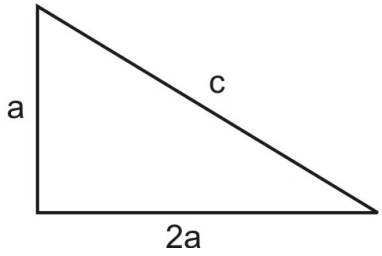
The area of the shaded region is therefore $2\pi - 4 = 2.28$.

Example C

In a right triangle, one leg is twice as long as the other and the perimeter is 28. What are the measures of the sides of the triangle?

Solution

Make a sketch and define variables:



Let: $a =$ length of the short leg

$2a =$ length of the long leg

$c =$ length of the hypotenuse

Write formulas:

The sides of the triangle are related in two different ways.

The perimeter is 28, so $a+2a+c=28 \Rightarrow 3a+c=28$

The triangle is a right triangle, so the measures of the sides must satisfy the Pythagorean Theorem:

$$a^2+(2a)^2=c^2 \Rightarrow a^2+4a^2=c^2 \Rightarrow 5a^2=c^2$$

or $$c=a\sqrt{5}=2.236a$$

Solve the equation:

Plug the value of c we just obtained into the perimeter equation: $3a+c=28$

$$3a+2.236a=28 \Rightarrow 5.236a=28 \Rightarrow a=5.35$$

The short leg is: $a=5.35$

The long leg is: $2a=10.70$

The hypotenuse is: $c=11.95$

Check: The legs of the triangle should satisfy the Pythagorean Theorem:

$a^2+b^2=5.35^2+10.70^2=143.1, c^2=11.95^2=142.80$. The results are approximately the same.

The perimeter of the triangle should be 28:

$a+b+c=5.35+10.70+11.95=28$.**The answer checks out.**

Watch this video for help with the Examples above.

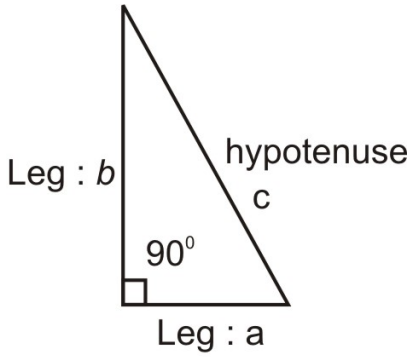
MEDIA

Click image to the left for more content.

CK-12 Foundation: Applications Using the Pythagorean Theorem

Vocabulary

- The **Pythagorean Theorem** is a statement of how the lengths of the sides of a right triangle are related to each other. A right triangle is one that contains a 90 degree angle. The side of the triangle opposite the 90 degree angle is called the **hypotenuse** and the sides of the triangle adjacent to the 90 degree angle are called the **legs**.



- If we let a and b represent the legs of the right triangle and c represent the hypotenuse then the **Pythagorean Theorem** can be stated as:

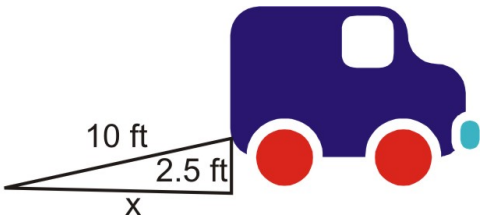
In a right triangle, the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the legs. That is: $a^2 + b^2 = c^2$.

Guided Practice

Mike is loading a moving van by walking up a ramp. The ramp is 10 feet long and the bed of the van is 2.5 feet above the ground. How far does the ramp extend past the back of the van?

Solution

Make a sketch:



Define variables: Let $x =$ how far the ramp extends past the back of the van.

Write a formula: Use the Pythagorean Theorem: $x^2 + 2.5^2 = 10^2$

Solve the equation:

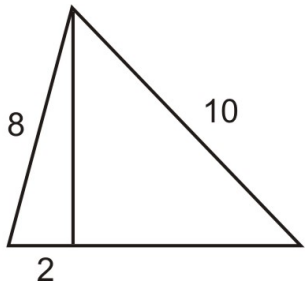
$$x^2 + 6.25 = 100$$
$$x^2 = 93.5$$
$$x = \sqrt{93.5} = 9.7 \ ft$$

Check by plugging the result in the Pythagorean Theorem:

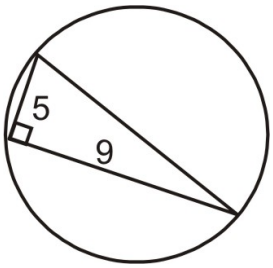
$9.7^2 + 2.5^2 = 94.09 + 6.25 = 100.34 \approx 100$. So the ramp is 10 feet long. **The answer checks out.**

Practice

1. In order to make a ramp that is $3ft$ high and covers $4ft$ of ground, how long must the ramp be?
2. A regulation baseball diamond is a square with 90 feet between bases. How far is second base from home plate?
3. Emanuel has a cardboard box that measures 20 cm long $\times$ 10 cm wide $\times$ 8 cm deep.
   a. What is the length of the diagonal across the bottom of the box?
   b. What is the length of the diagonal from a bottom corner to the opposite top corner?
4. Samuel places a ladder against his house. The base of the ladder is 6 feet from the house and the ladder is 10 feet long.
   a. How high above the ground does the ladder touch the wall of the house?
   b. If the edge of the roof is 10 feet off the ground and sticks out 1.5 feet beyond the wall, how far is it from the edge of the roof to the top of the ladder?
5. Find the area of the triangle below if the area of a triangle is defined as $A = \frac{1}{2}\ base \times height$:



6. Instead of walking along the two sides of a rectangular field, Mario decided to cut across the diagonal. He thus saves a distance that is half of the long side of the field.
   a. Find the length of the long side of the field given that the short side is 123 feet.
   b. Find the length of the diagonal.
7. Marcus sails due north and Sandra sails due east from the same starting point. In two hours Marcus' boat is 35 miles from the starting point and Sandra's boat is 28 miles from the starting point.
   a. How far are the boats from each other?
   b. Sandra then sails 21 miles due north while Marcus stays put. How far is Sandra from the original starting point?
   c. How far is Sandra from Marcus now?
8. Determine the area of the circle below. (Hint: the hypotenuse of the triangle is the diameter of the circle.)



9. A rectangle's length is $1in$ longer than its width and if the diagonal has a length of $29in$, what are the lengths of the sides of the rectangle?
10. For an isosceles triangle with sides of the length given, find the length of hypotenuse:
    a. 1
    b. 2
    c. 3
    d. n

1.4 Special Right Triangles

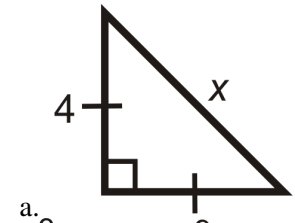
Learning Objectives

- Identify and use the ratios involved with isosceles right triangles.
- Identify and use the ratios involved with 30-60-90 triangles.

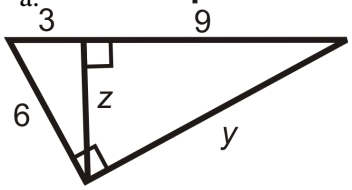
Review Queue

Find the value of the missing variable(s). Simplify all radicals.

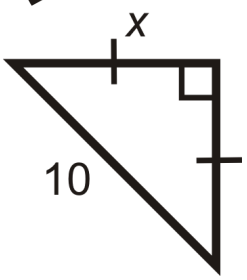
a.



b.



c.

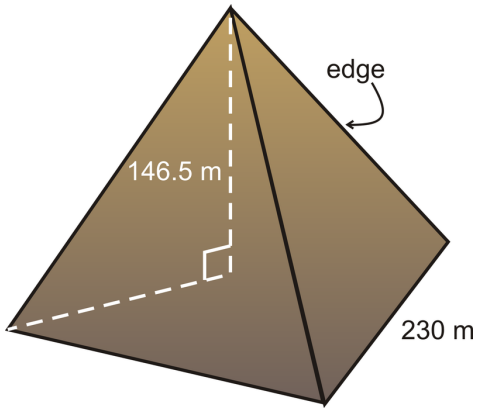


d. Do the lengths 6, 6, and $6\sqrt{2}$ make a right triangle?
e. Do the lengths $3, 3\sqrt{3}$, and 6 make a right triangle?

Know What? The Great Giza Pyramid is a pyramid with a square base and four isosceles triangles that meet at a point. It is thought that the original height was 146.5 meters and the base edges were 230 meters.

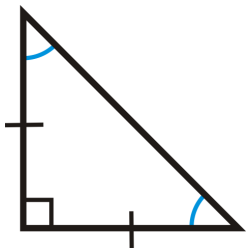
First, find the length of the edge of the isosceles triangles. Then, determine if the isosceles triangles are also equilateral triangles. Round your answers to the nearest tenth.

You can assume that the height of the pyramid is from the center of the square base and is a vertical line.



Isosceles Right Triangles

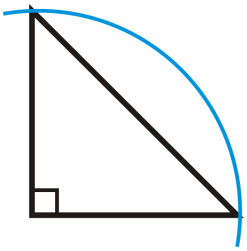
There are two types of special right triangles, based on their angle measures. The first is an isosceles right triangle. Here, the legs are congruent and, by the Base Angles Theorem, the base angles will also be congruent. Therefore, the angle measures will be $90°, 45°$, and $45°$. You will also hear an isosceles right triangle called a 45-45-90 triangle. Because the three angles are always the same, all isosceles right triangles are similar.



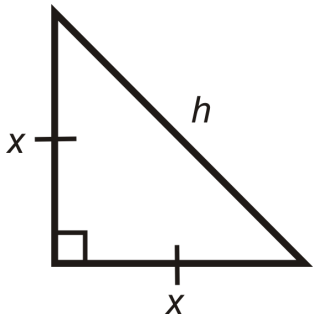
Investigation 8-2: Properties of an Isosceles Right Triangle

Tools Needed: Pencil, paper, compass, ruler, protractor

a. Construct an isosceles right triangle with 2 in legs. Use the SAS construction that you learned in Chapter 4.



b. Find the measure of the hypotenuse. What is it? Simplify the radical.
c. Now, let's say the legs are of length x and the hypotenuse is h. Use the Pythagorean Theorem to find the hypotenuse. What is it? How is this similar to your answer in #2?



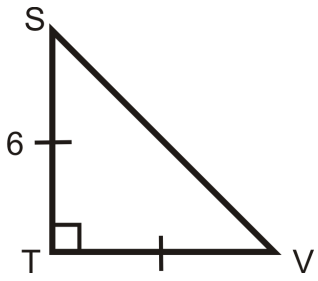
$$x^2 + x^2 = h^2$$
$$2x^2 = h^2$$
$$x\sqrt{2} = h$$

45-45-90 Corollary: If a triangle is an isosceles right triangle, then its sides are in the extended ratio $x : x : x\sqrt{2}$.

Step 3 in the above investigation proves the 45-45-90 Triangle Theorem. So, anytime you have a right triangle with congruent legs or congruent angles, then the sides will always be in the ratio $x : x : x\sqrt{2}$. The hypotenuse is always $x\sqrt{2}$ because that is the longest length. This is a specific case of the Pythagorean Theorem, so it will still work, if for some reason you forget this corollary.

Example 1: Find the length of the missing sides.

a)



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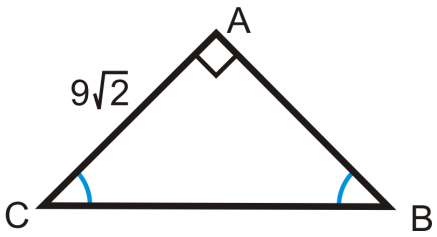


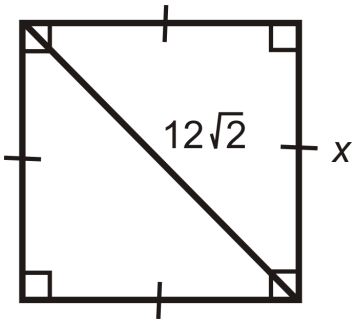
Solution: Use the $x : x : x\sqrt{2}$ ratio.

a) $TV = 6$ because it is equal to ST. So, $SV = 6\sqrt{2}$.

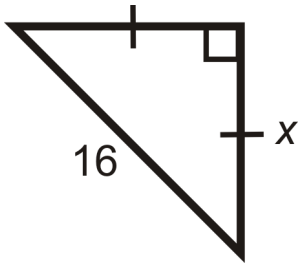
b) $AB = 9\sqrt{2}$ because it is equal to AC. So, $BC = 9\sqrt{2} \cdot \sqrt{2} = 9 \cdot 2 = 18$.

Example 2: Find the length of x.

a)



b)



Solution: Again, use the $x : x : x\sqrt{2}$ ratio, but in these two we are given the hypotenuse. We need to solve for x in the ratio.

a) $12\sqrt{2} = x\sqrt{2}$
$12 = x$

b) $x\sqrt{2} = 16$
$x = \frac{16}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{16\sqrt{2}}{2} = 8\sqrt{2}$

In part b, we ***rationalized the denominator***. Whenever there is a radical in the denominator of a fraction, multiply the top and bottom by that radical. This will cancel out the radical from the denominator and reduce the fraction.

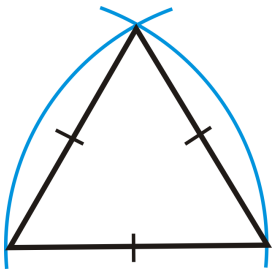
30-60-90 Triangles

The second special right triangle is called a 30-60-90 triangle, after the three angles. To construct a 30-60-90 triangle, start with an equilateral triangle.

Investigation 8-3: Properties of a 30-60-90 Triangle

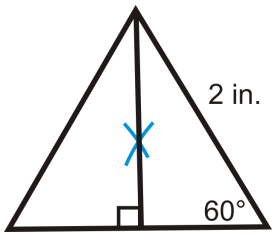
Tools Needed: Pencil, paper, ruler, compass

1. Construct an equilateral triangle with 2 in sides.



2. Draw or construct the altitude from the top vertex to the base for two congruent triangles.

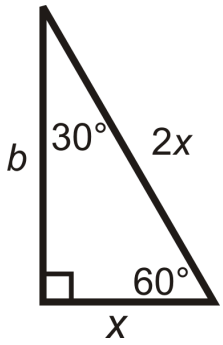
3. Find the measure of the two angles at the top vertex and the length of the shorter leg.



The top angles are each 30° *and the shorter leg is 1 in because the altitude of an equilateral triangle is also the angle and perpendicular bisector.*

4. Find the length of the longer leg, using the Pythagorean Theorem. Simplify the radical.

5. Now, let's say the shorter leg is length x and the hypotenuse is $2x$. Use the Pythagorean Theorem to find the longer leg. What is it? How is this similar to your answer in #4?



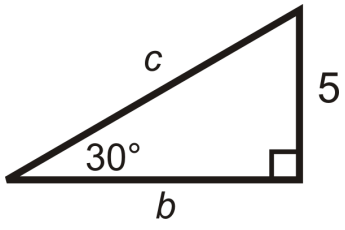
$$x^2 + b^2 = (2x)^2$$
$$x^2 + b^2 = 4x^2$$
$$b^2 = 3x^2$$
$$b = x\sqrt{3}$$

30-60-90 Corollary: If a triangle is a 30-60-90 triangle, then its sides are in the extended ratio $x : x\sqrt{3} : 2x$.

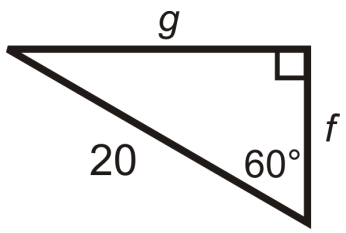
Step 5 in the above investigation proves the 30-60-90 Corollary. The shortest leg is always x, the longest leg is always $x\sqrt{3}$, and the hypotenuse is always $2x$. If you ever forget this corollary, then you can still use the Pythagorean Theorem.

Example 3: Find the length of the missing sides.

a)



b)



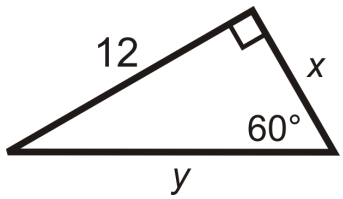
Solution: In part a, we are given the shortest leg and in part b, we are given the hypotenuse.

a) If $x = 5$, then the longer leg, $b = 5\sqrt{3}$, and the hypotenuse, $c = 2(5) = 10$.

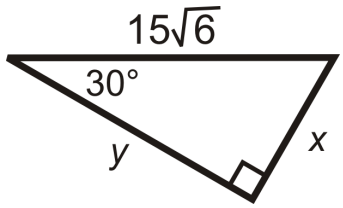
b) Now, $2x = 20$, so the shorter leg, $f = 10$, and the longer leg, $g = 10\sqrt{3}$.

Example 4: Find the value of x and y.

a)



b)



Solution: In part a, we are given the longer leg and in part b, we are given the hypotenuse.

a) $x\sqrt{3} = 12$

$x = \frac{12}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{12\sqrt{3}}{3} = 4\sqrt{3}$

Then, the hypotenuse would be

$y = 2\left(4\sqrt{3}\right) = 8\sqrt{3}$

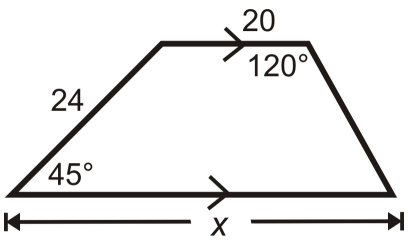
b) $2x = 15\sqrt{6}$

$x = \frac{15\sqrt{6}}{2}$

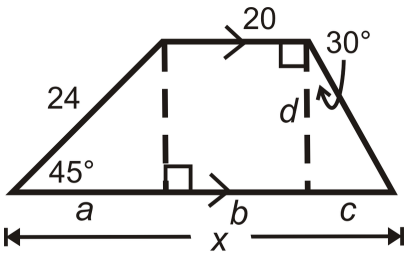
The, the longer leg would be

$y = \left(\frac{15\sqrt{6}}{2}\right) \cdot \sqrt{3} = \frac{15\sqrt{18}}{2} = \frac{45\sqrt{2}}{2}$

Example 5: Find the measure of x.



Solution: Think of this trapezoid as a rectangle, between a 45-45-90 triangle and a 30-60-90 triangle.



From this picture, $x = a + b + c$. First, find a, which is a leg of an isosceles right triangle.

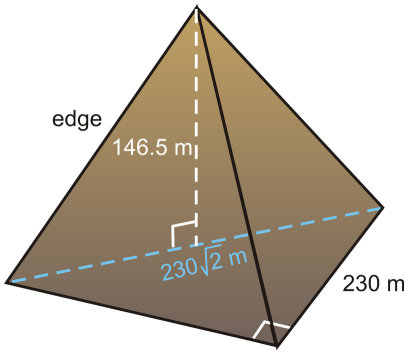
$$a = \frac{24}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{24\sqrt{2}}{2} = 12\sqrt{2}$$

$a = d$, so we can use this to find c, which is the shorter leg of a 30-60-90 triangle.

$$c = \frac{12\sqrt{2}}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{12\sqrt{6}}{3} = 4\sqrt{6}$$

$b = 20$, so $x = 12\sqrt{2} + 20 + 4\sqrt{6}$. Nothing simplifies, so this is how we leave our answer.

Know What? Revisited The line that the vertical height is perpendicular to is the diagonal of the square base. This length (blue) is the same as the hypotenuse of an isosceles right triangle because half of a square is an isosceles right triangle. So, the diagonal is $230\sqrt{2}$. Therefore, the base of the right triangle with 146.5 as the leg is half of $230\sqrt{2}$ or $115\sqrt{2}$. Do the Pythagorean Theorem to find the edge.



$$edge = \sqrt{\left(115\sqrt{2}\right)^2 + 146.5^2} \approx 218.9\ m$$

In order for the sides to be equilateral triangles, this length should be 230 meters. It is not, so the triangles are isosceles.

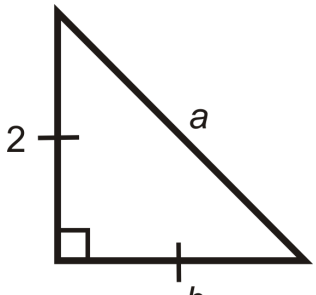
Review Questions

1. In an isosceles right triangle, if a leg is x, then the hypotenuse is _____.

2. In a 30-60-90 triangle, if the shorter leg is x, then the longer leg is _____ and the hypotenuse is _____-_____.
3. A square has sides of length 15. What is the length of the diagonal?
4. A square's diagonal is 22. What is the length of each side?
5. A rectangle has sides of length 4 and $4\sqrt{3}$. What is the length of the diagonal?
6. A baseball diamond is a square with 90 foot sides. What is the distance from home base to second base? (HINT: It's the length of the diagonal).

For questions 7-18, find the lengths of the missing sides.

7.



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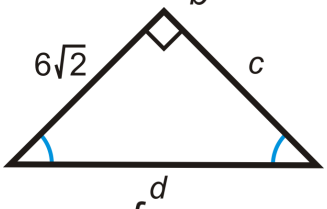

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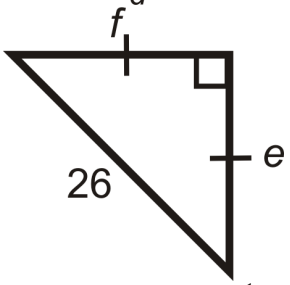

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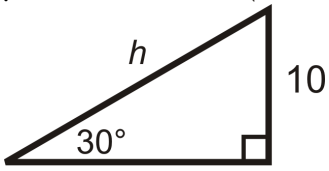

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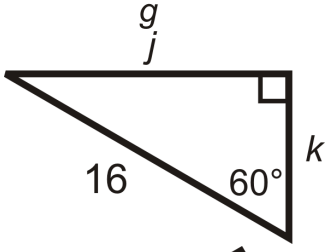

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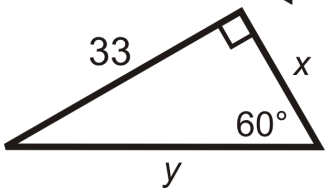

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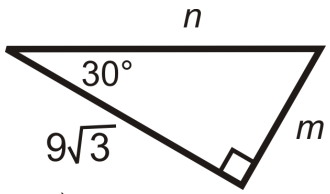

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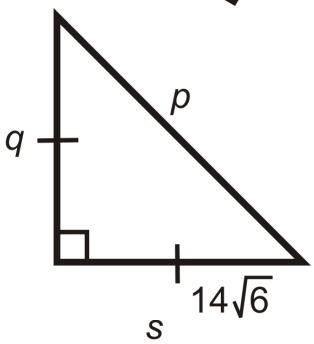

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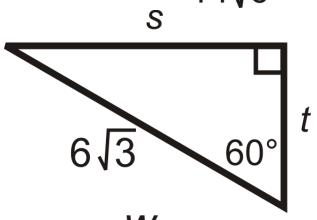

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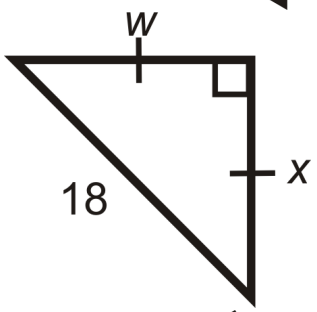

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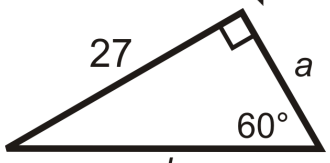

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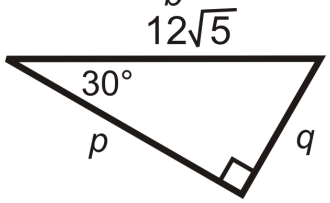
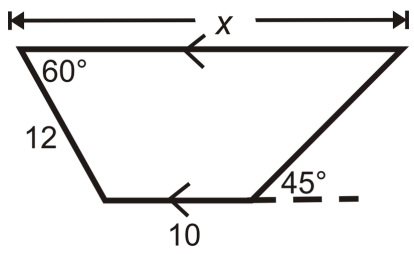
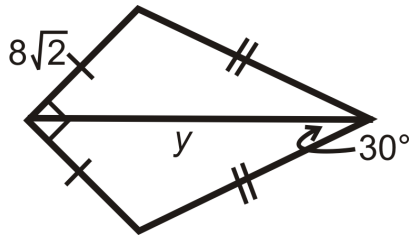


19. Do the lengths $8\sqrt{2}, 8\sqrt{6}$, *and* $16\sqrt{2}$ make a special right triangle? If so, which one?
20. Do the lengths $4\sqrt{3}, 4\sqrt{6}$ *and* $8\sqrt{3}$ make a special right triangle? If so, which one?
21. Find the measure of x.



22. Find the measure of y.



23. What is the ratio of the sides of a rectangle if the diagonal divides the rectangle into two 30-60-90 triangles?
24. What is the length of the sides of a square with diagonal 8 in?

For questions 25-28, it might be helpful to recall #25 from section 8.1.

25. What is the height of an equilateral triangle with sides of length 3 in?
26. What is the area of an equilateral triangle with sides of length 5 ft?
27. A regular hexagon has sides of length 3 in. What is the area of the hexagon? (*Hint: the hexagon is made up a 6 equilateral triangles.*)
28. The area of an equilateral triangle is $36\sqrt{3}$. What is the length of a side?
29. If a road has a grade of 30°, this means that its angle of elevation is 30°. If you travel 1.5 miles on this road, how much elevation have you gained in feet (5280 ft = 1 mile)?
30. Four isosceles triangles are formed when both diagonals are drawn in a square. If the length of each side in the square is s, what are the lengths of the legs of the isosceles triangles?

Review Queue Answers

a. $4^2+4^2=x^2$
$32=x^2$
$x=4\sqrt{2}$

b. $3^2+z^2=6^2$
$z^2=27$
$z=3\sqrt{3}$

$\left(3\sqrt{3}\right)^2+9^2=y^2$
$108=y^2$
$y=6\sqrt{3}$

c. $x^2+x^2=10^2$
$2x^2=100$
$x^2=50$
$x=5\sqrt{2}$

d. Yes, $6^2+6^2=\left(6\sqrt{2}\right)^2\rightarrow 36+36=72$

e. Yes, $3^2+\left(3\sqrt{3}\right)^2=6^2\rightarrow 9+27=36$

1.5 Sine, Cosine, Tangent

Here you'll learn what the three trigonometric ratios are and how to find their value for a right triangle's non-right angle.

What if you were given a right triangle and told that its sides measure 3, 4, and 5 inches? How could you find the sine, cosine, and tangent of one of the triangle's non-right angles? After completing this Concept, you'll be able to solve for these trigonometric ratios.

Watch This

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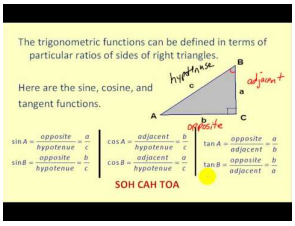
Click image to the left for more content.

CK-12 Foundation: Chapter8SoneCosineTangentA

Watch the parts of the video dealing with the sine, cosine, and tangent.



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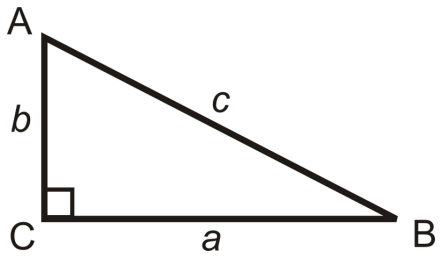
James Sousa: Introduction to Trigonometric Functions Using Triangles

Guidance

The word trigonometry comes from two words meaning *triangle* and *measure*. In this lesson we will define three trigonometric (or trig) functions.

Trigonometry: The study of the relationships between the sides and angles of right triangles.

In trigonometry, sides are named in reference to a particular angle. The hypotenuse of a triangle is always the same, but the terms **adjacent** and **opposite** depend on which angle you are referencing. A side adjacent to an angle is the leg of the triangle that helps form the angle. A side opposite to an angle is the leg of the triangle that does not help form the angle. We never reference the right angle when referring to trig ratios.



The three basic trig ratios are called, sine, cosine and tangent. At this point, we will only take the sine, cosine and tangent of acute angles. However, you will learn that you can use these ratios with obtuse angles as well.

To explore the relationship between the unit circle, the Pythagorean Theorem, and the trigonometric ratios, click here.

Sine Ratio: For an acute angle x in a right triangle, the $\sin x$ is equal to the ratio of the side opposite the angle over the hypotenuse of the triangle. Using the triangle above, $\sin A = \frac{a}{c}$ and $\sin B = \frac{b}{c}$.

Cosine Ratio: For an acute angle x in a right triangle, the $\cos x$ is equal to the ratio of the side adjacent to the angle over the hypotenuse of the triangle.Using the triangle above, $\cos A = \frac{b}{c}$ and $\cos B = \frac{a}{c}$.

Tangent Ratio: For an acute angle x, in a right triangle, the $\tan x$ is equal to the ratio of the side opposite to the angle over the side adjacent to x. Using the triangle above, $\tan A = \frac{a}{b}$ and $\tan B = \frac{b}{a}$.

There are a few important things to note about the way we write these ratios. First, keep in mind that the abbreviations $\sin x, \cos x$, and $\tan x$ are all functions. Second, be careful when using the abbreviations that you still pronounce the full name of each function. When we write $\sin x$ it is still pronounced *sine,* with a long "*i*". When we write $\cos x$, we still say co-sine. And when we write $\tan x$, we still say tangent. An easy way to remember ratios is to use the pneumonic SOH-CAH-TOA.

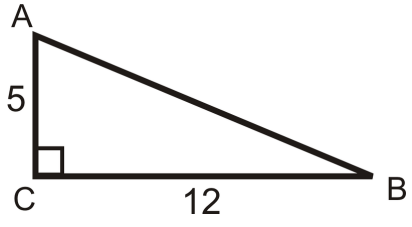
$$\text{Sine} = \frac{\text{Opposite}}{\text{Hypotenuse}} \qquad \text{Cosine} = \frac{\text{Adjacent}}{\text{Hypotenuse}} \qquad \text{Tangent} = \frac{\text{Opposite}}{\text{Adjacent}}$$

A few important points:

- Always reduce ratios when you can.
- Use the Pythagorean Theorem to find the missing side (if there is one).
- The tangent ratio can be bigger than 1 (the other two cannot).
- If two right triangles are similar, then their sine, cosine, and tangent ratios will be the same (because they will reduce to the same ratio).
- If there is a radical in the denominator, rationalize the denominator.
- The sine, cosine and tangent for an angle are fixed.

Example A

Find the sine, cosine and tangent ratios of $\angle A$.



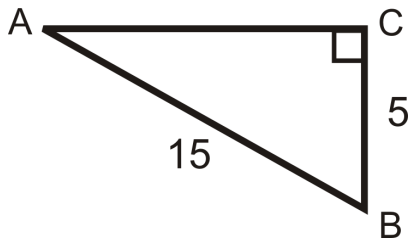
First, we need to use the Pythagorean Theorem to find the length of the hypotenuse.

$$5^2 + 12^2 = h^2$$
$$13 = h$$

So, $\sin A = \frac{12}{13}, \cos A = \frac{5}{13}$, and $\tan A = \frac{12}{5}$.

Example B

Find the sine, cosine, and tangent of $\angle B$.



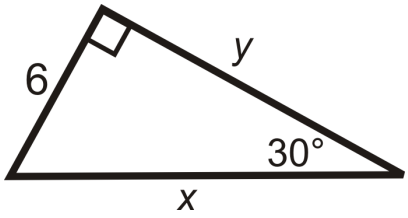
Find the length of the missing side.

$$AC^2 + 5^2 = 15^2$$
$$AC^2 = 200$$
$$AC = 10\sqrt{2}$$

Therefore, $\sin B = \frac{10\sqrt{2}}{15} = \frac{2\sqrt{2}}{3}, \cos B = \frac{5}{15} = \frac{1}{3}$, and $\tan B = \frac{10\sqrt{2}}{5} = 2\sqrt{2}$.

Example C

Find the sine, cosine and tangent of 30°.



This is a special right triangle, a 30-60-90 triangle. So, if the short leg is 6, then the long leg is $6\sqrt{3}$ and the hypotenuse is 12.

$\sin 30^\circ = \frac{6}{12} = \frac{1}{2}, \cos 30^\circ = \frac{6\sqrt{3}}{12} = \frac{\sqrt{3}}{2}$, and $\tan 30^\circ = \frac{6}{6\sqrt{3}} = \frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3}$.

Watch this video for help with the Examples above.

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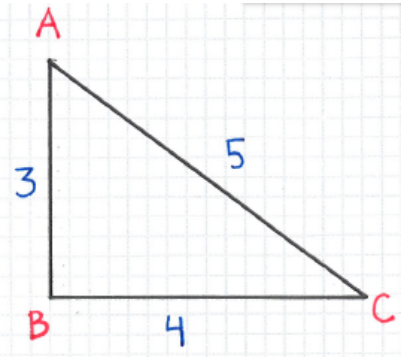
CK-12 Foundation: Chapter8SineCosineTangentB

Concept Problem Revisited



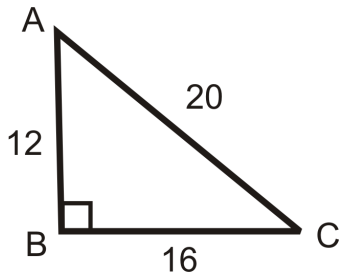
The trigonometric ratios for the non-right angles in the triangle above are:
$\sin A = \frac{4}{5}, \cos A = \frac{3}{5}, \tan A = \frac{4}{3}, \sin B = \frac{3}{5}, \cos B = \frac{4}{5}$, and $\tan B = \frac{3}{4}$.

Vocabulary

Trigonometry is the study of the relationships between the sides and angles of right triangles. The legs are called ***adjacent*** or ***opposite*** depending on which ***acute*** angle is being used. The three trigonometric (or trig) ratios are ***sine***, ***cosine***, and ***tangent***.

Guided Practice

Answer the questions about the following image. Reduce all fractions.



1. What is $\sin A$?
2. What is $\cos A$?
3. What is $\tan A$?

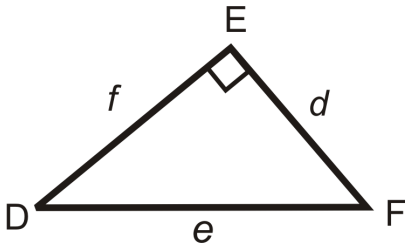
Answers:

1. $\sin A = \frac{16}{20} = \frac{4}{5}$

2. $\cos A = \frac{12}{20} = \frac{3}{5}$
3. $\tan A = \frac{16}{12} = \frac{4}{3}$

Practice

Use the diagram to fill in the blanks below.



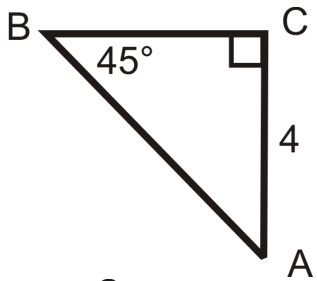
1. $\tan D = \frac{?}{?}$
2. $\sin F = \frac{?}{?}$
3. $\tan F = \frac{?}{?}$
4. $\cos F = \frac{?}{?}$
5. $\sin D = \frac{?}{?}$
6. $\cos D = \frac{?}{?}$

From questions 1-6, we can conclude the following. Fill in the blanks.

7. $\cos$ ___ $= \sin F$ and $\sin$ ___ $= \cos F$.
8. $\tan D$ and $\tan F$ are _____ of each other.

Find the sine, cosine and tangent of $\angle A$. Reduce all fractions and radicals.

9.



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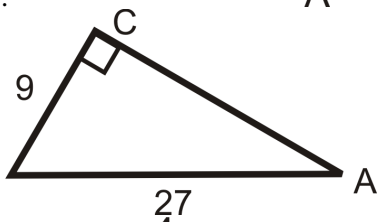

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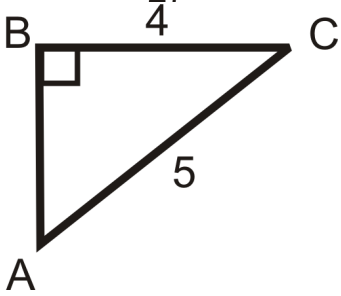
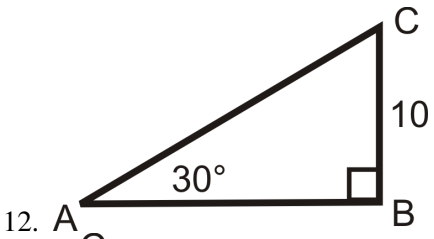
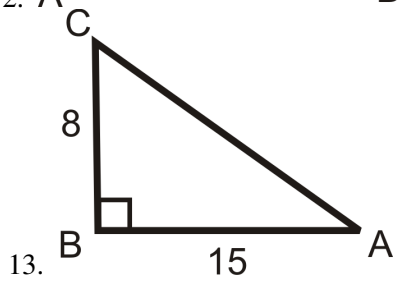


12. 

13. 

14. Explain why the sine of an angle will never be greater than 1.
15. Explain why the tangent of a 45° angle will always be 1.
16. As the degree of an angle increases, will the tangent of the angle increase or decrease? Explain.

1.6 Trigonometric Ratios with a Calculator

Here you'll learn how to solve for missing sides in right triangles that are not one of the special right triangles.

What if you wanted to find the missing sides of a right triangle with angles of 20° and 70° and a hypotenuse length of 10 inches? How could you use trigonometry to help you? After completing this Concept, you'll be able to solve problems like this one.

Watch This

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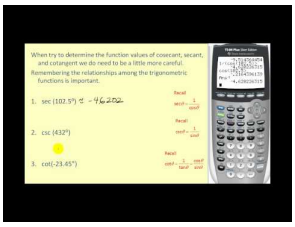
CK-12 Foundation: Chapter8TrigonometricRatioswithaCalculatorA



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James Sousa: Determining Trigonometric Function Values on the Calculator

Guidance

The trigonometric ratios are not dependent on the exact side lengths, but the angles. There is one fixed value for every angle, from 0° to 90°. Your scientific (or graphing) calculator knows the values of the sine, cosine and tangent of all of these angles. Depending on your calculator, you should have [SIN], [COS], and [TAN] buttons. Use these to find the sine, cosine, and tangent of any acute angle. One application of the trigonometric ratios is to use them to find the missing sides of a right triangle. All you need is one angle, other than the right angle, and one side.

Example A

Find the trigonometric value, using your calculator. Round to 4 decimal places.

a) $\sin 78°$

b) $\cos 60°$

c) $\tan 15°$

Depending on your calculator, you enter the degree and then press the trig button or the other way around. Also, make sure the mode of your calculator is in *DEGREES*.

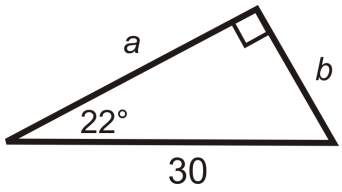
a) $\sin 78^\circ = 0.97815$

b) $\cos 60^\circ = 0.5$

c) $\tan 15^\circ = 0.26795$

Example B

Find the value of each variable. Round your answer to the nearest tenth.

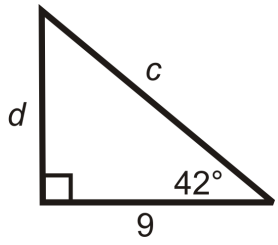


We are given the hypotenuse. Use *sine* to find b, and *cosine* to find a. Use your calculator to evaluate the sine and cosine of the angles.

$$\sin 22^\circ = \frac{b}{30} \qquad \cos 22^\circ = \frac{a}{30}$$
$$30 \cdot \sin 22^\circ = b \qquad 30 \cdot \cos 22^\circ = a$$
$$b \approx 11.2 \qquad a \approx 27.8$$

Example C

Find the value of each variable. Round your answer to the nearest tenth.



We are given the adjacent leg to 42°. To find c, use *cosine* and use *tangent* to find d.

$$\cos 42^\circ = \frac{adjacent}{hypotenuse} = \frac{9}{c} \qquad \tan 42^\circ = \frac{opposite}{adjacent} = \frac{d}{9}$$
$$c \cdot \cos 42^\circ = 9 \qquad 9 \cdot \tan 42^\circ = d$$
$$c = \frac{9}{\cos 42^\circ} \approx 12.1 \qquad d \approx 27.0$$

Any time you use trigonometric ratios, use only the information that you are given. This will result in the most accurate answers.

Watch this video for help with the Examples above.

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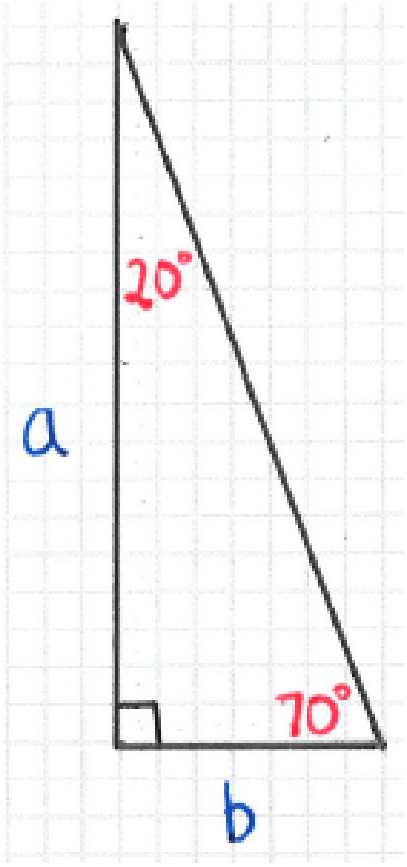
CK-12 Foundation: Chapter8TrigonometricRatioswithaCalculatorB

Concept Problem Revisited

Use trigonometric ratios to find the missing sides. Round to the nearest tenth.



Find the length of a and b using sine or cosine ratios:

$$\cos 20^\circ = \frac{a}{10}$$
$$10 \cdot \cos 20^\circ = a$$
$$a \approx 9.4$$

$$\sin 70^\circ = \frac{a}{10}$$
$$10 \cdot \sin 70^\circ = a$$
$$a \approx 9.4$$

$$\sin 20^\circ = \frac{b}{10}$$
$$10 \cdot \sin 20^\circ = b$$
$$b \approx 3.4$$

$$\cos 70^\circ = \frac{b}{10}$$
$$10 \cdot \cos 70^\circ = b$$
$$b \approx 3.4$$

What do you notice about the relationship between sine and cosine of complementary angles? When dealing with a pair of complementary angles, the sine of one angle is equal to the cosine of the other angle. If the $10\cos 20^\circ \approx 9.4$ then you can compute $10\sin 20^\circ \approx 9.4$ without your calculator because 20° and 70° are complementary angles. Click on the following links for more information on how to use this special relationship:

Video One Video Two

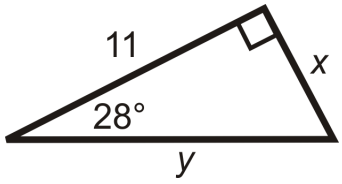
Vocabulary

Trigonometry is the study of the relationships between the sides and angles of right triangles. The legs are called ***adjacent*** or ***opposite*** depending on which ***acute*** angle is being used. The three trigonometric (or trig) ratios are ***sine***, ***cosine***, and ***tangent***.

Guided Practice

1. What is $\tan 45^\circ$?

2. Find the length of the missing sides and round your answers to the nearest tenth:



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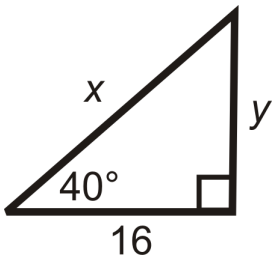
3. Find the length of the missing sides and round your answers to the nearest tenth:



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Answers:

1. Using your calculator, you should find that $\tan 45^\circ = 1$?

2. Use tangent for x and cosine for y.

$$\tan 28^\circ = \frac{x}{11} \qquad \cos 28^\circ = \frac{11}{y}$$

$$11 \cdot \tan 28^\circ = x \qquad \frac{11}{\cos 28^\circ} = y$$

$$x \approx 5.8 \qquad y \approx 12.5$$

3. Use tangent for y and cosine for x.

$$\tan 40^\circ = \frac{y}{16} \qquad \cos 40^\circ = \frac{16}{x}$$

$$16 \cdot \tan 40^\circ = y \qquad \frac{16}{\cos 40^\circ} = x$$

$$y \approx 13.4 \qquad x \approx 20.9$$

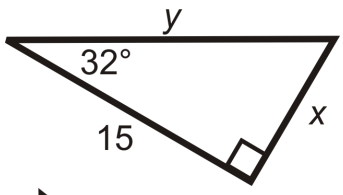
Practice

Use your calculator to find the value of each trig function below. Round to four decimal places.

1. $\sin 24^\circ$
2. $\cos 45^\circ$
3. $\tan 88^\circ$
4. $\sin 43^\circ$
5. $\tan 12^\circ$
6. $\cos 79^\circ$
7. $\sin 82^\circ$

Find the length of the missing sides. Round your answers to the nearest tenth.

8.



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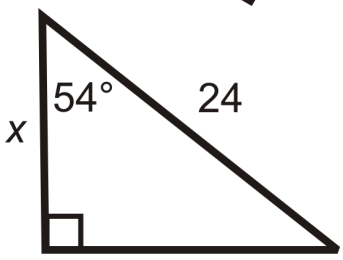

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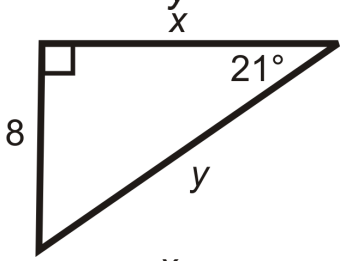

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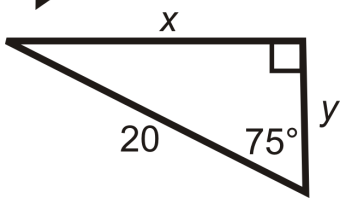


12. Find x. $\sin 35^\circ = \cos x^\circ$
13. Find x. $\cos 0^\circ = \sin x^\circ$
14. Solve for the unknown. $\sin(x - 10^\circ) = \cos(35^\circ)$
15. Simplify but don't solve. $\sin 60^\circ + \cos 30^\circ$

16. Find $\sin 80^\circ$ and $\cos 10^\circ$.
17. Use your knowledge of where the trigonometric ratios come from to explain your result to the previous question.
18. Generalize your result to the previous two questions. If $\sin\theta = x$, then $\cos ? = x$.
19. Find m if $\sin(8m) = \cos(4m-6)$
20. Find m if $\sin(10m-2) = \cos(5m+2)$.

1.7 Inverse Trigonometric Ratios

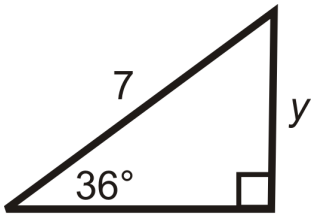
Learning Objectives

- Use the inverse trigonometric ratios to find an angle in a right triangle.
- Solve a right triangle.
- Apply inverse trigonometric ratios to real-life situation and special right triangles.

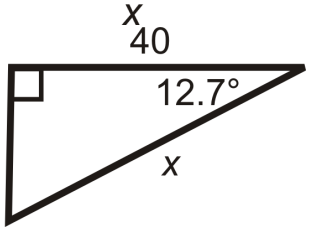
Review Queue

Find the lengths of the missing sides. Round your answer to the nearest hundredth.

a.



b.



c. Draw an isosceles right triangle with legs of length 3. What is the hypotenuse?
d. Use the triangle from #3, to find the sine, cosine, and tangent of 45°.
e. Explain why $\tan 45^\circ = 1$.

Know What? The longest escalator in North America is at the Wheaton Metro Station in Maryland. It is 230 feet long and is 115 ft high. What is the angle of elevation, x°, of this escalator?

Inverse Trigonometric Ratios

The word ***inverse*** is probably familiar to you. In mathematics, once you learn how to do an operation, you also learn how to "undo" it. For example, you may remember that addition and subtraction are considered inverse operations. Multiplication and division are also inverse operations. In algebra you used inverse operations to solve equations and inequalities.

When we apply the word inverse to the trigonometric ratios, we can find the acute angle measures within a right triangle. Normally, if you are given an angle and a side of a right triangle, you can find the other two sides, using sine, cosine or tangent. With the inverse trig ratios, you can find the angle measure, given two sides.

Inverse Tangent: If you know the opposite side and adjacent side of an angle in a right triangle, you can use inverse tangent to find the measure of the angle.

Inverse tangent is also called arctangent and is labeled $\tan^{-1}$ or *arctan*. The "-1" indicates inverse.

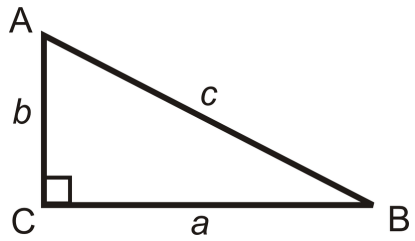
Inverse Sine: If you know the opposite side of an angle and the hypotenuse in a right triangle, you can use inverse sine to find the measure of the angle.

Inverse sine is also called arcsine and is labeled $\sin^{-1}$ or *arcsin*.

Inverse Cosine: If you know the adjacent side of an angle and the hypotenuse in a right triangle, you can use inverse cosine to find the measure of the angle.

Inverse cosine is also called arccosine and is labeled $\cos^{-1}$ or *arccos*.

Using the triangle below, the inverse trigonometric ratios look like this:



$$\tan^{-1}\left(\frac{b}{a}\right) = m\angle B \qquad \tan^{-1}\left(\frac{a}{b}\right) = m\angle A$$

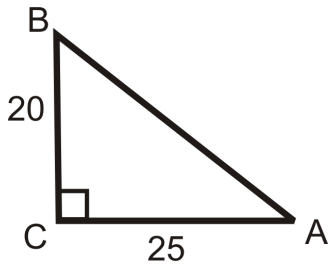
$$\sin^{-1}\left(\frac{b}{c}\right) = m\angle B \qquad \sin^{-1}\left(\frac{a}{c}\right) = m\angle A$$

$$\cos^{-1}\left(\frac{a}{c}\right) = m\angle B \qquad \cos^{-1}\left(\frac{b}{c}\right) = m\angle A$$

In order to actually find the measure of the angles, you will need you use your calculator. On most scientific and graphing calculators, the buttons look like $[\text{SIN}^{-1}]$, $[\text{COS}^{-1}]$, and $[\text{TAN}^{-1}]$. Typically, you might have to hit a shift or 2^{nd} button to access these functions. For example, on the TI-83 and 84, $[2^{nd}][\text{SIN}]$ is $[\text{SIN}^{-1}]$. Again, make sure the mode is in degrees.

When you find the inverse of a trigonometric function, you put the word *arc* in front of it. So, the inverse of a tangent is called the arctangent (or arctan for short). Think of the arctangent as a tool you can use like any other inverse operation when solving a problem. If tangent tells you the ratio of the lengths of the sides opposite and adjacent to an angle, then tangent inverse tells you the measure of an angle with a given ratio.

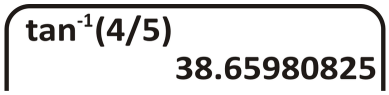
Example 1: Use the sides of the triangle and your calculator to find the value of $\angle A$. Round your answer to the nearest tenth of a degree.



Solution: In reference to $\angle A$, we are given the *opposite* leg and the *adjacent* leg. This means we should use the *tangent* ratio.

$\tan A = \frac{20}{25} = \frac{4}{5}$, therefore $\tan^{-1}\left(\frac{4}{5}\right) = m\angle A$. Use your calculator.

If you are using a TI-83 or 84, the keystrokes would be: $[2^{nd}][\text{TAN}]\left(\frac{4}{5}\right)$**[ENTER]** and the screen looks like:



So, $m\angle A = 38.7°$

Example 2: $\angle A$ is an acute angle in a right triangle. Use your calculator to find $m\angle A$ to the nearest tenth of a degree.

a) $\sin A = 0.68$

b) $\cos A = 0.85$

c) $\tan A = 0.34$

Solution:

a) $m\angle A = \sin^{-1} 0.68 = 42.8°$

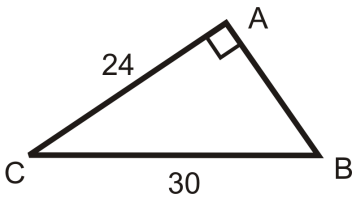
b) $m\angle A = \cos^{-1} 0.85 = 31.8°$

c) $m\angle A = \tan^{-1} 0.34 = 18.8°$

Solving Triangles

Now that we know how to use inverse trigonometric ratios to find the measure of the acute angles in a right triangle, we can solve right triangles. To solve a right triangle, you would need to find all sides and angles in a right triangle, using any method. When solving a right triangle, you could use sine, cosine or tangent, inverse sine, inverse cosine, or inverse tangent, or the Pythagorean Theorem. Remember when solving right triangles to only use the values that you are given.

Example 3: Solve the right triangle.



Solution: To solve this right triangle, we need to find $AB, m\angle C$ and $m\angle B$. Use AC and CB to give the most accurate answers.

$\underline{AB}$: Use the Pythagorean Theorem.

$$\begin{aligned} 24^2 + AB^2 &= 30^2 \\ 576 + AB^2 &= 900 \\ AB^2 &= 324 \\ AB &= \sqrt{324} = 18 \end{aligned}$$

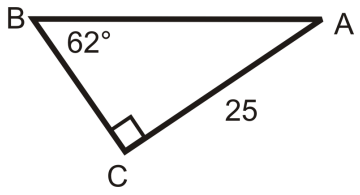
$\underline{m\angle B}$: Use the inverse sine ratio.

$$\begin{aligned} \sin B &= \frac{24}{30} = \frac{4}{5} \\ \sin^{-1}\left(\frac{4}{5}\right) &= 53.1^\circ = m\angle B \end{aligned}$$

$\underline{m\angle C}$: Use the inverse cosine ratio.

$$\begin{aligned} \cos C &= \frac{24}{30} = \frac{4}{5} \\ \cos^{-1}\left(\frac{4}{5}\right) &= 36.9^\circ = m\angle C \end{aligned}$$

Example 4: Solve the right triangle.



Solution: To solve this right triangle, we need to find AB, BC and $m\angle A$.

$\underline{AB}$: Use sine ratio.

$$\begin{aligned} \sin 62^\circ &= \frac{25}{AB} \\ AB &= \frac{25}{\sin 62^\circ} \\ AB &\approx 28.31 \end{aligned}$$

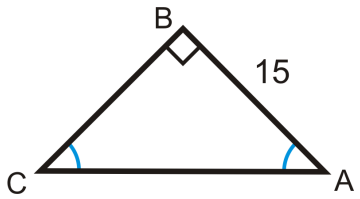
$\underline{BC}$: Use tangent ratio.

$$\begin{aligned} \tan 62^\circ &= \frac{25}{BC} \\ BC &= \frac{25}{\tan 62^\circ} \\ BC &\approx 13.30 \end{aligned}$$

$m\angle A$: Use Triangle Sum Theorem

$$62^\circ + 90^\circ + m\angle A = 180^\circ$$
$$m\angle A = 28^\circ$$

Example 5: Solve the right triangle.



Solution: Even though, there are no angle measures given, we know that the two acute angles are congruent, making them both 45°. Therefore, this is a 45-45-90 triangle. You can use the trigonometric ratios or the special right triangle ratios.

Trigonometric Ratios

$$\tan 45^\circ = \frac{15}{BC}$$
$$BC = \frac{15}{\tan 45^\circ} = 15$$

$$\sin 45^\circ = \frac{15}{AC}$$
$$AC = \frac{15}{\sin 45^\circ} \approx 21.21$$

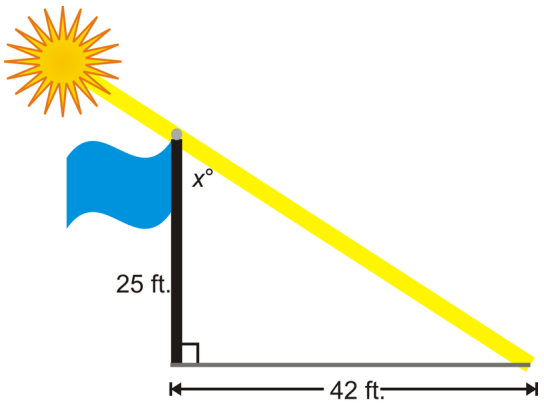
45-45-90 Triangle Ratios

$$BC = AB = 15, AC = 15\sqrt{2} \approx 21.21$$

Real-Life Situations

Much like the trigonometric ratios, the inverse trig ratios can be used in several real-life situations. Here are a couple examples.

Example 6: A 25 foot tall flagpole casts a 42 feet shadow. What is the angle that the sun hits the flagpole?

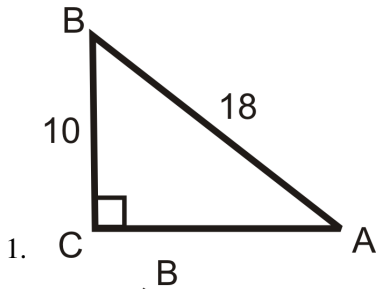


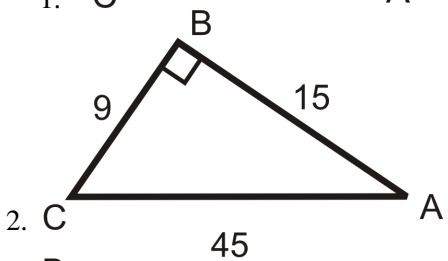
Solution: First, draw a picture. The angle that the sun hits the flagpole is the acute angle at the top of the triangle, x°. From the picture, we can see that we need to use the inverse tangent ratio.

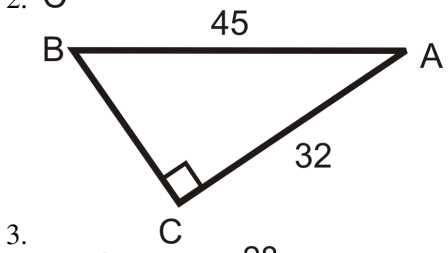
$$\tan x = \frac{42}{25}$$
$$\tan^{-1} \frac{42}{25} \approx 59.2^\circ = x$$

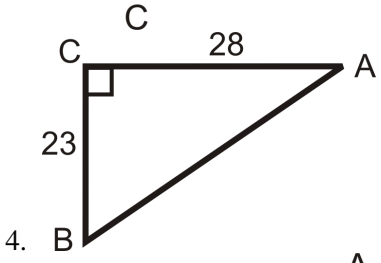
Review Questions

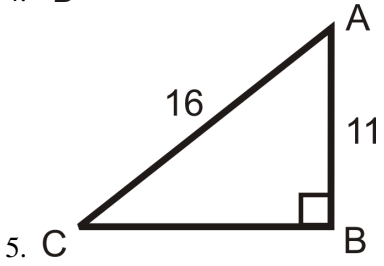
Use your calculator to find $m\angle A$ to the nearest tenth of a degree.

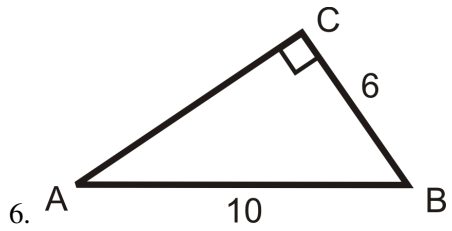
1. 

2. 

3. 

4. 

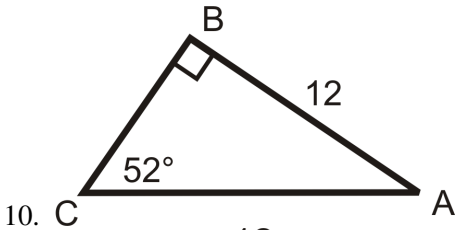
5. 

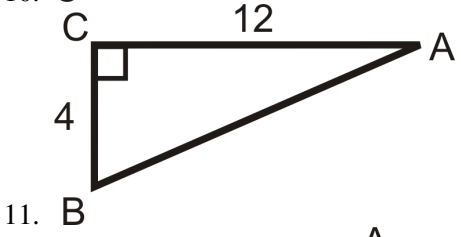
6. 

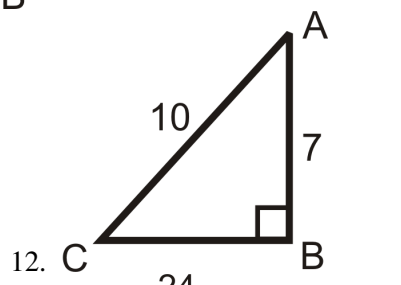
Let $\angle A$ be an acute angle in a right triangle. Find $m\angle A$ to the nearest tenth of a degree.

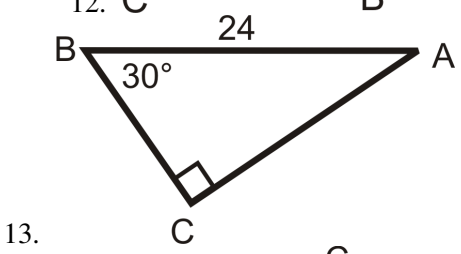
7. $\sin A = 0.5684$
8. $\cos A = 0.1234$
9. $\tan A = 2.78$

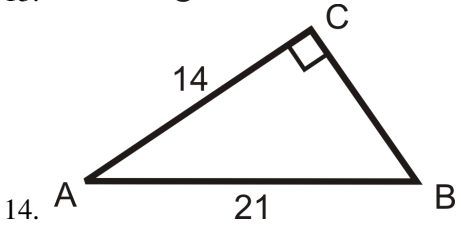
Solving the following right triangles. Find all missing sides and angles.

10. 

11. 

12. 

13. 

14. 

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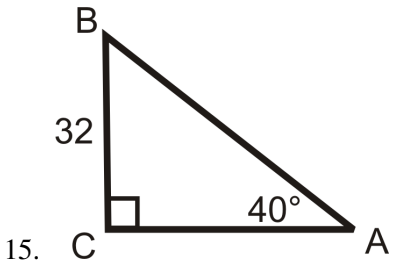
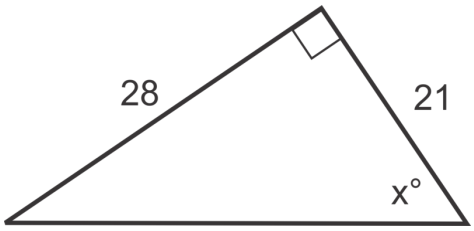


16. ***Writing*** Explain when to use a trigonometric ratio to find a side length of a right triangle and when to use the Pythagorean Theorem.

Real-Life Situations Use what you know about right triangles to solve for the missing angle. If needed, draw a picture. Round all answers to the nearest tenth of a degree.

17. A 75 foot building casts an 82 foot shadow. What is the angle that the sun hits the building?
18. Elizabeth wants to know the angle at which the sun hits a tree in her backyard at 3 pm. She finds that the length of the tree's shadow is 24 ft at 3 pm. At the same time of day, her shadow is 6 ft 5 inches. If Elizabeth is 4 ft 8 inches tall, find the height of the tree and hence the angle at which the sunlight hits the tree.
19. Tommy was solving the triangle below and made a mistake. What did he do wrong?



$$\tan^{-1}\left(\frac{21}{28}\right) \approx 36.9^\circ$$

20. Tommy then continued the problem and set up the equation: $\cos 36.9^\circ = \frac{21}{h}$. By solving this equation he found that the hypotenuse was 26.3 units. Did he use the correct trigonometric ratio here? Is his answer correct? Why or why not?
21. How could Tommy have found the hypotenuse in the triangle another way and avoided making his mistake?

Examining Patterns Below is a table that shows the sine, cosine, and tangent values for eight different angle measures. Answer the following questions.

TABLE 1.1:

	10°	20°	30°	40°	50°	60°	70°	80°
Sine	0.1736	0.3420	0.5	0.6428	0.7660	0.8660	0.9397	0.9848
Cosine	0.9848	0.9397	0.8660	0.7660	0.6428	0.5	0.3420	0.1736
Tangent	0.1763	0.3640	0.5774	0.8391	1.1918	1.7321	2.7475	5.6713

26. What value is equal to $\sin 40^\circ$?
27. What value is equal to $\cos 70^\circ$?
28. Describe what happens to the sine values as the angle measures increase.
29. Describe what happens to the cosine values as the angle measures increase.
30. What two numbers are the sine and cosine values between?
31. Find $\tan 85^\circ, \tan 89^\circ$, and $\tan 89.5^\circ$ using your calculator. Now, describe what happens to the tangent values as the angle measures increase.

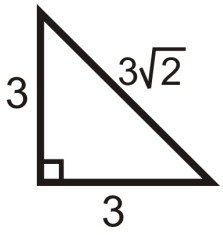
32. Explain why all of the sine and cosine values are less than one. (hint: think about the sides in the triangle and the relationships between their lengths)

Review Queue Answers

a. $\sin 36^\circ = \frac{y}{7}$ $\cos 36^\circ = \frac{x}{7}$
$y = 4.11$ $x = 5.66$

b. $\cos 12.7^\circ = \frac{40}{x}$ $\tan 12.7^\circ = \frac{y}{40}$
$x = 41.00$ $y = 9.01$

c.



d. $\sin 45^\circ = \frac{3}{3\sqrt{2}} = \frac{\sqrt{2}}{2}$
$\cos 45^\circ = \frac{3}{3\sqrt{2}} = \frac{\sqrt{2}}{2}$
$\tan 45^\circ = \frac{3}{3} = 1$

e. The tangent of 45° equals one because it is the ratio of the opposite side over the adjacent side. In an isosceles right triangle, or 45-45-90 triangle, the opposite and adjacent sides are the same, making the ratio always 1.

1.8 Application Problems

Here you'll use the Pythagorean Theorem and trigonometric ratios to solve the real world application problems.

A 40-foot-tall tree casts a shadow of 80 feet. What is the angle of elevation from the end of the shadow to the top of the tree with respect to the ground?

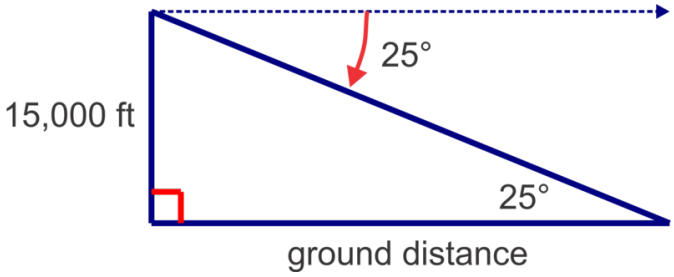
Guidance

When solving word problems, it is important to understand the terminology used to describe angles. In trigonometric problems, the terms angle of elevation and angle of depression are commonly used. Both of these angles are always measured from a horizontal line as shown in the diagrams below.

Example A

An airplane approaching an airport spots the runway at an angle of depression of 25°. If the airplane is 15,000 ft above the ground, how far (ground distance) is the plane from the runway? Give your answer to the nearest 100 ft.

Solution: Make a diagram to illustrate the situation described and then use a trigonometric ratio to solve. Keep in mind that an angle of depression is down from a horizontal line of sight-in this case a horizontal line from the pilot of the plane parallel to the ground.



Note that the angle of depression and the alternate interior angle will be congruent, so the angle in the triangle is also 25°.

From the picture, we can see that we should use the tangent ratio to find the ground distance.

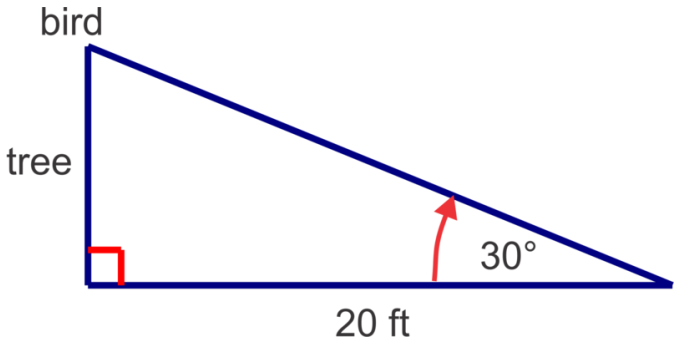
$$\tan 25^\circ = \frac{15000}{d}$$

$$d = \frac{15000}{\tan 25^\circ} \approx 32,200 \ ft$$

Example B

Rachel spots a bird in a tree at an angle of elevation of 30°. If Rachel is 20 ft from the base of the tree, how high up in the tree is the bird? Give your answer to the nearest tenth of a foot.

Solution: Make a diagram to illustrate the situation. Keep in mind that there will be a right triangle and that the right angle is formed by the ground and the trunk of the tree.



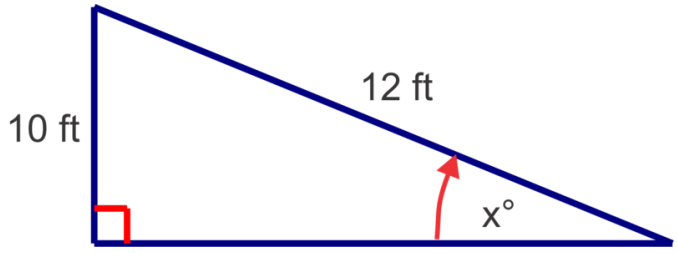
Here we can use the tangent ratio to solve for the height of the bird

$$\tan 30^\circ = \frac{h}{20}$$
$$h = 20 \tan 30^\circ \approx 11.5 \ ft$$

Example C

A 12 ft ladder is leaning against a house and reaches 10 ft up the side of the house. To the nearest degree, what angle does the ladder make with the ground?

Solution: In this problem, we will need to find an angle. By making a sketch of the triangle we can see which inverse trigonometric ratio to use.



$$\sin x^\circ = \frac{10}{12}$$
$$\sin^{-1}\left(\frac{10}{12}\right) \approx 56^\circ$$

Concept Problem Revisit If you draw this situation, you will see that we are dealing with a right triangle. The side opposite the angle of elevation is 40. The side adjacent to the angle is 80. Therefore, we can use the tangent to find the angle of elevation.

$$\tan x^\circ = \frac{40}{80} = \frac{1}{2}$$
$$\tan^{-1}\left(\frac{1}{2}\right) = \approx 26.57^\circ$$

Guided Practice

Use a trigonometry to solve the following application problems.

1. A ramp makes a 20° angle with the ground. If door the ramp leads to is 2 ft above the ground, how long is the ramp? Give your answer to the nearest tenth of a foot.

2. Charlie lets out 90 ft of kite string. If the angle of elevation of the string is 70°, approximately how high is the kite? Give your answer to the nearest foot.

3. A ship's sonar spots a wreckage at an angle of depression of 32°. If the depth of the ocean is about 250 ft, how far is the wreckage (measured along the surface of the water) from the ship, to the nearest foot.

Answers

1.

$$\sin 20^\circ = \frac{2}{x}$$
$$x = \frac{2}{\sin 20^\circ} \approx 5.8 \ ft$$

2.

$$\sin 70^\circ = \frac{x}{90}$$
$$x = 90 \sin 70^\circ \approx 85 \ ft$$

3.

$$\tan 32^\circ = \frac{250}{x}$$
$$x = \frac{250}{\tan 32^\circ} \approx 400 \ ft$$

Vocabulary

Angle of Elevation
An angle measured up from a horizontal line.

Angle of Depression
An angle measured down from a horizontal line.

Practice

Use the Pythagorean Theorem and/or trigonometry to solve the following word problems.

1. A square has sides of length 8 inches. To the nearest tenth of an inch, what is the length of its diagonal?
2. Layne spots a sailboat from her fifth floor balcony, about 25 m above the beach, at an angle of depression of 3°. To the nearest meter, how far out is the boat?
3. A zip line takes passengers on a 200 m ride from high up in the trees to a ground level platform. If the angle of elevation of the zip line is 10°, how high above ground is the tree top start platform? Give your answer to the nearest meter.
4. The angle of depression from the top of an apartment building to the base of a fountain in a nearby park is 57°. If the building is 150 ft tall, how far away, to the nearest foot, is the fountain?
5. A playground slide platform is 6 ft above ground. If the slide is 8 ft long and the end of the slide is 1 ft above ground, what angle does the slide make with the ground? Give your answer to the nearest degree.
6. Benjamin spots a tree directly across the river from where he is standing. He then walks 27 ft upstream and determines that the angle between his previous position and the tree on the other side of the river is 73°. How wide, to the nearest foot, is the river?
7. A rectangle has sides of length 6 in and 10 in. To the nearest degree, what angle does the diagonal make with the longer side?
8. Tommy is flying his kite one afternoon and notices that he has let out the entire 130 ft of string. The angle his string makes with the ground is 48°. How high, to the nearest foot, is his kite at this time?
9. A tree struck by lightning in a storm breaks and falls over to form a triangle with the ground. The tip of the tree makes a 18° angle with the ground 21 ft from the base of the tree. What was the height of the tree to the nearest foot?
10. Upon descent an airplane is 19,000 ft above the ground. The air traffic control tower is 190 ft tall. It is determined that the angle of elevation from the top of the tower to the plane is 15°. To the nearest mile, find the ground distance from the airplane to the tower.
11. Why will the sine and cosine ratios always be less than 1?

CHAPTER 2 Quadrilaterals

Chapter Outline

2.1 PARALLELOGRAMS

2.2 QUADRILATERALS THAT ARE PARALLELOGRAMS

2.3 QUADRILATERALS

2.4 QUADRILATERAL CLASSIFICATION

2.5 REFERENCES

2.1 Parallelograms

Here you'll learn what a parallelogram is and how apply theorems about its sides, angles, and diagonals to solve for unknown values.

What if you were told that $FGHI$ is a parallelogram and you are given the length of FG and the measure of $\angle F$? What can you determine about HI, $\angle H$, $\angle G$, and $\angle I$? After completing this Concept, you'll be able to apply parallelogram theorems to answer such questions.

Watch This

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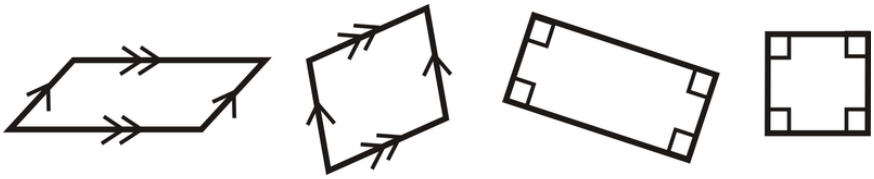
CK-12 Parallelograms

Guidance

A **parallelogram** is a quadrilateral with two pairs of parallel sides.

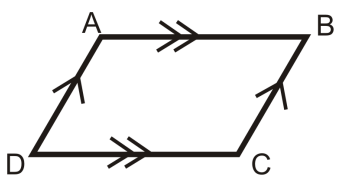


Notice that each pair of sides is marked parallel (for the last two shapes, remember that when two lines are perpendicular to the same line then they are parallel). Parallelograms have a lot of interesting properties. (You can explore the properties of a parallelogram at: http://www.mathwarehouse.com/geometry/quadrilaterals/parallelograms/interactive-parallelogram.php)

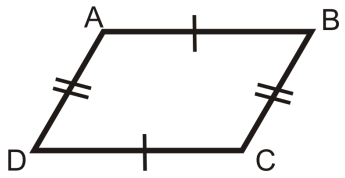
Facts about Parallelograms

1) **Opposite Sides Theorem:** If a quadrilateral is a parallelogram, then both pairs of opposite sides are congruent.

If

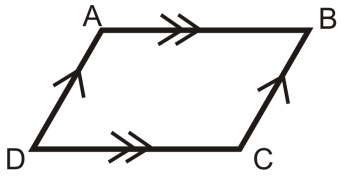


then

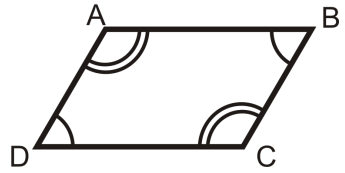


2) **Opposite Angles Theorem:** If a quadrilateral is a parallelogram, then both pairs of opposite angles are congruent.

If

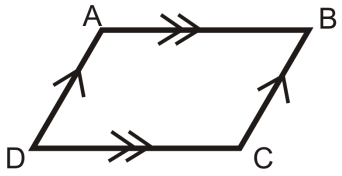


then



3) **Consecutive Angles Theorem:** If a quadrilateral is a parallelogram, then all pairs of consecutive angles are supplementary.

If



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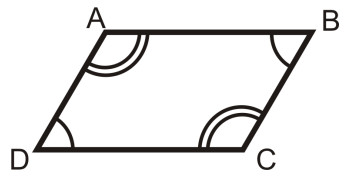
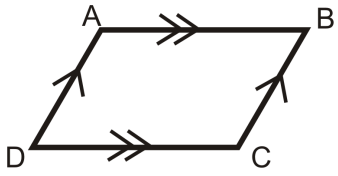


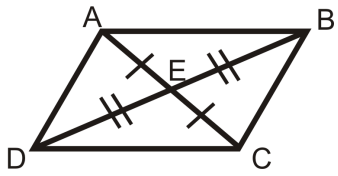
$$m\angle A + m\angle D = 180^\circ$$
$$m\angle A + m\angle B = 180^\circ$$
$$m\angle B + m\angle C = 180^\circ$$
$$m\angle C + m\angle D = 180^\circ$$

4) **Parallelogram Diagonals Theorem:** If a quadrilateral is a parallelogram, then the diagonals bisect each other.

If



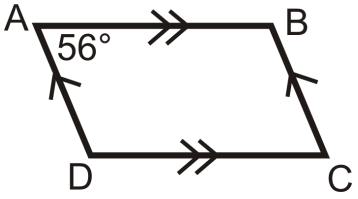
then



Example A

ABCD is a parallelogram. If $m\angle A = 56^\circ$, find the measure of the other angles.

First draw a picture. When labeling the vertices, the letters are listed, in order.

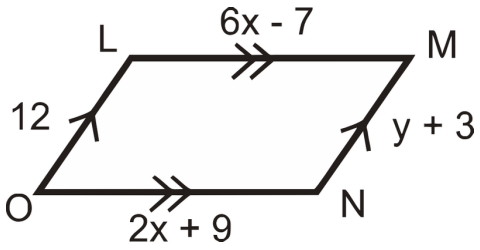


If $m\angle A = 56^\circ$, then $m\angle C = 56^\circ$ by the Opposite Angles Theorem.

$$\begin{aligned} m\angle A + m\angle B &= 180^\circ \quad \text{by the Consecutive Angles Theorem.} \\ 56^\circ + m\angle B &= 180^\circ \\ m\angle B &= 124^\circ \quad m\angle D = 124^\circ \quad \text{because it is an opposite angle to } \angle B. \end{aligned}$$

Example B

Find the values of x and y.

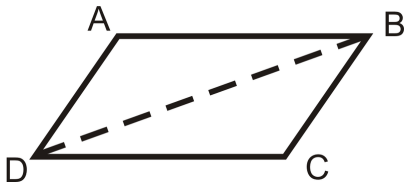


Remember that opposite sides of a parallelogram are congruent. Set up equations and solve.

$$\begin{aligned} 6x - 7 &= 2x + 9 \\ 4x &= 16 \\ x &= 4 \end{aligned} \qquad \begin{aligned} y + 3 &= 12 \\ y &= 9 \end{aligned}$$

Example C

Prove the Opposite Sides Theorem.



Given: $ABCD$ is a parallelogram with diagonal $\overline{BD}$

Prove: $\overline{AB} \cong \overline{DC}, \overline{AD} \cong \overline{BC}$

TABLE 2.1:

Statement	*Reason*
1. $ABCD$ is a parallelogram with diagonal $\overline{BD}$	1. Given
2. $\overline{AB} \| \overline{DC}, \overline{AD} \| \overline{BC}$	2. Definition of a parallelogram
3. $\angle ABD \cong \angle BDC, \angle ADB \cong \angle DBC$	3. Alternate Interior Angles Theorem
4. $\overline{DB} \cong \overline{DB}$	4. Reflexive PoC
5. $\triangle ABD \cong \triangle CDB$	5. ASA
6. $\overline{AB} \cong \overline{DC}, \overline{AD} \cong \overline{BC}$	6. CPCTC

The proof of the Opposite Angles Theorem is almost identical. You will try this proof in the problem set.

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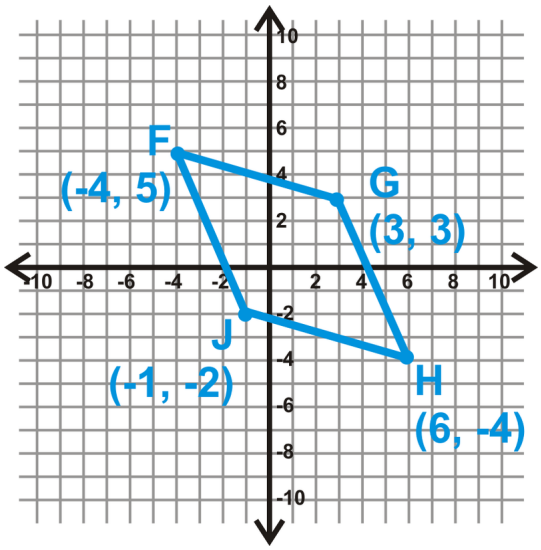
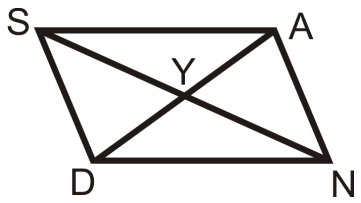
CK-12 Parallelograms

Guided Practice

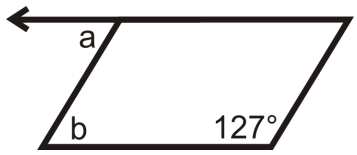
1. Show that the diagonals of $FGHJ$ bisect each other.



2. *SAND* is a parallelogram, $SY = 4x - 11$ and $YN = x + 10$. Solve for x.



3. Find the measures of a and b in the parallelogram below:



Answers:

1. Find the midpoint of each diagonal.

$$\text{Midpoint of } \overline{FH}: \quad \left(\frac{-4+6}{2}, \frac{5-4}{2}\right) = (1, 0.5)$$

$$\text{Midpoint of } \overline{GJ}: \quad \left(\frac{3-1}{2}, \frac{3-2}{2}\right) = (1, 0.5)$$

Because they are the same point, the diagonals intersect at each other's midpoint. This means they bisect each other.

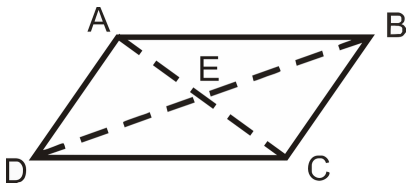
2. Because this is a parallelogram, the diagonals bisect each other and $SY \cong YN$.

$$\begin{aligned} SY &= YN \\ 4x - 11 &= x + 10 \\ 3x &= 21 \\ x &= 7 \end{aligned}$$

3. Consecutive angles are supplementary so $127^\circ + m\angle b = 180^\circ$ which means that $m\angle b = 53^\circ$. a and b are alternate interior angles and since the lines are parallel (since its a parallelogram), that means that $m\angle a = m\angle b = 53^\circ$.

Practice

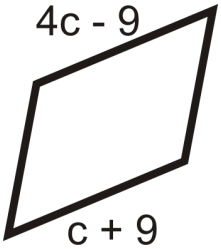
ABCD is a parallelogram. Fill in the blanks below.



1. If $AB = 6$, then $CD =$ _____.
2. If $AE = 4$, then $AC =$ _____.
3. If $m\angle ADC = 80°, m\angle DAB =$ _____.
4. If $m\angle BAC = 45°, m\angle ACD =$ _____.
5. If $m\angle CBD = 62°, m\angle ADB =$ _____.
6. If $DB = 16$, then $DE =$ _____.
7. If $m\angle B = 72°$ in parallelogram *ABCD*, find the other three angles.
8. If $m\angle S = 143°$ in parallelogram *PQRS*, find the other three angles.
9. If $\overline{AB} \perp \overline{BC}$ in parallelogram *ABCD*, find the measure of all four angles.
10. If $m\angle F = x°$ in parallelogram *EFGH*, find the other three angles.

For questions 11-18, find the values of the variable(s). All the figures below are parallelograms.

11.



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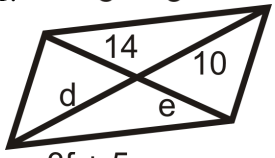

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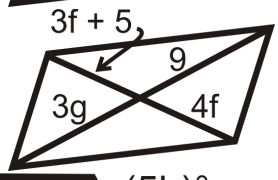

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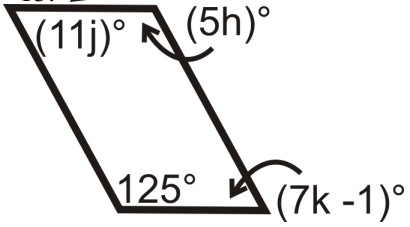

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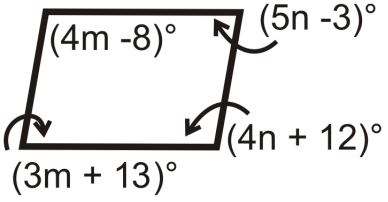

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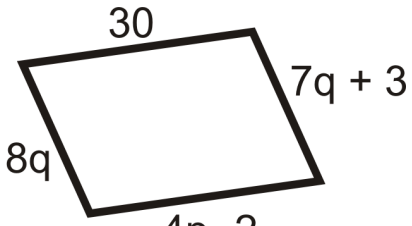

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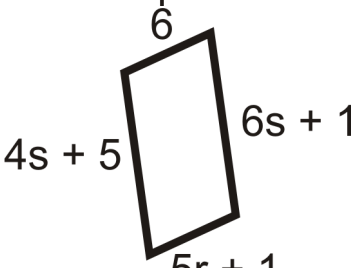
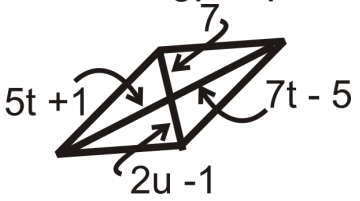
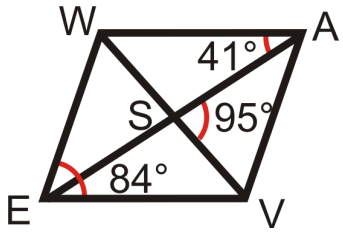


18.



Use the parallelogram $WAVE$ to find:



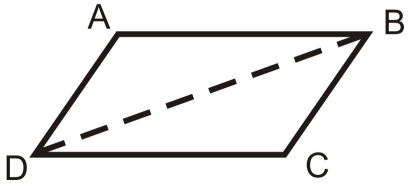
19. $m\angle AWE$
20. $m\angle ESV$
21. $m\angle WEA$
22. $m\angle AVW$

Find the point of intersection of the diagonals to see if $EFGH$ is a parallelogram.

23. $E(-1,3), F(3,4), G(5,-1), H(1,-2)$
24. $E(3,-2), F(7,0), G(9,-4), H(5,-4)$
25. $E(-6,3), F(2,5), G(6,-3), H(-4,-5)$
26. $E(-2,-2), F(-4,-6), G(-6,-4), H(-4,0)$

Fill in the blanks in the proofs below.

27. ***Opposite Angles Theorem***



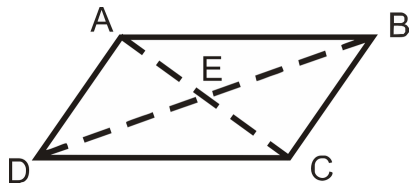
Given: $ABCD$ is a parallelogram with diagonal $\overline{BD}$

Prove: $\angle A \cong \angle C$

TABLE 2.2:

Statement	***Reason***
1.	1. Given
2. $\overline{AB} \| \overline{DC}, \overline{AD} \| \overline{BC}$	2.
3.	3. Alternate Interior Angles Theorem
4.	4. Reflexive PoC
5. $\triangle ABD \cong \triangle CDB$	5.
6. $\angle A \cong \angle C$	6.

28. ***Parallelogram Diagonals Theorem***



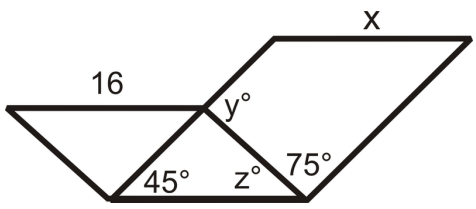
Given: $ABCD$ is a parallelogram with diagonals $\overline{BD}$ and $\overline{AC}$

Prove: $\overline{AE} \cong \overline{EC}, \overline{DE} \cong \overline{EB}$

TABLE 2.3:

Statement	***Reason***
1.	1.
2.	2. Definition of a parallelogram
3.	3. Alternate Interior Angles Theorem
4. $\overline{AB} \cong \overline{DC}$	4.
5.	5.
6. $\overline{AE} \cong \overline{EC}, \overline{DE} \cong \overline{EB}$	6.

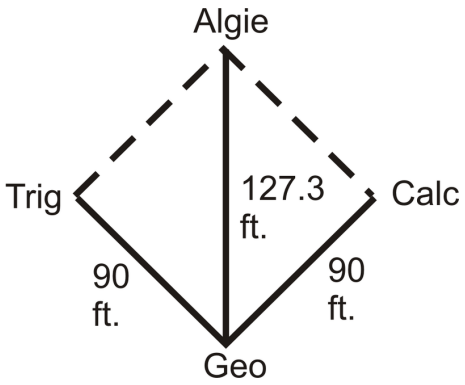
29. Find x, y°, and z°. (The two quadrilaterals with the same side are parallelograms.)



2.2 Quadrilaterals that are Parallelograms

Here you'll learn how to prove that a quadrilateral is a parallelogram.

What if four friends, Geo, Trig, Algie, and Calc were marking out a baseball diamond? Geo is standing at home plate. Trig is 90 feet away at 3^{rd} base, Algie is 127.3 feet away at 2^{nd} base, and Calc is 90 feet away at 1^{st} base. The angle at home plate is 90°, from 1^{st} to 3^{rd} is 90°. Find the length of the other diagonal and determine if the baseball diamond is a parallelogram. After completing this Concept, you'll be able to answer questions like this based on your knowledge of parallelograms.



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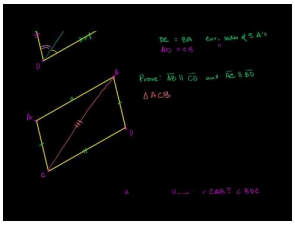
CK-12 Foundation: Chapter6QuadrilateralsthatareParallelogramsA



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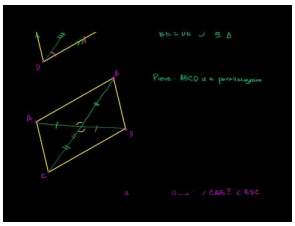
Khan Academy: Opposite Sides of a Parallelogram Congruent



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Khan Academy: Diagonals of a Parallelogram Bisect Each Other

Guidance

Recall that a parallelogram is a quadrilateral with two pairs of parallel sides. Even if a quadrilateral is not marked with having two pairs of sides, it still might be a parallelogram. The following is a list of theorems that will help you decide if a quadrilateral is a parallelogram or not.

Opposite Sides Theorem Converse: If the opposite sides of a quadrilateral are congruent, then the figure is a parallelogram.

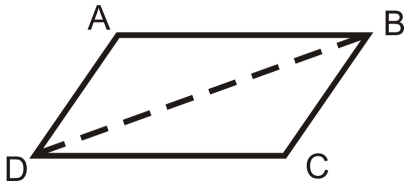
Opposite Angles Theorem Converse: If the opposite angles of a quadrilateral are congruent, then the figure is a parallelogram.

Parallelogram Diagonals Theorem Converse: If the diagonals of a quadrilateral bisect each other, then the figure is a parallelogram.

Theorem: If a quadrilateral has one set of parallel lines that are also congruent, then it is a parallelogram.

Each of these theorems can be a way to show that a quadrilateral is a parallelogram.

Proof of the Opposite Sides Theorem Converse:



Given: $\overline{AB} \cong \overline{DC}$, $\overline{AD} \cong \overline{BC}$

Prove: $ABCD$ is a parallelogram

TABLE 2.4:

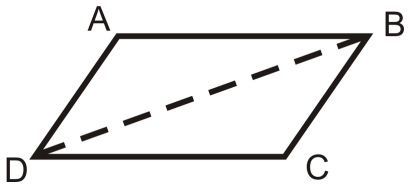
Statement	*Reason*
1. $\overline{AB} \cong \overline{DC}$, $\overline{AD} \cong \overline{BC}$	Given
2. $\overline{DB} \cong \overline{DB}$	Reflexive PoC
3. $\triangle ABD \cong \triangle CDB$	SSS
4. $\angle ABD \cong \angle BDC$, $\angle ADB \cong \angle DBC$	CPCTC
5. $\overline{AB} \parallel \overline{DC}$, $\overline{AD} \parallel \overline{BC}$	Alternate Interior Angles Converse
6. $ABCD$ is a parallelogram	Definition of a parallelogram

To show that a quadrilateral is a parallelogram in the $x-y$ plane, you will need to use a combination of the slope formulas, the distance formula and the midpoint formula. For example, to use the Definition of a Parallelogram, you would need to ***find the slope of all four sides*** to see if the opposite sides are parallel. To use the Opposite Sides Converse, you would have to find the length (***using the distance formula or the Pythagorean Theorem***) of each

side to see if the opposite sides are congruent. To use the Parallelogram Diagonals Converse, you would need to use the ***midpoint formula*** for each diagonal to see if the midpoint is the same for both. Finally, you can use the last Theorem in this Concept (that if one pair of opposite sides is both congruent and parallel then the quadrilateral is a parallelogram) in the coordinate plane. To use this theorem, you would need to show that one pair of opposite sides has the same slope (***slope formula***) and the same length (***distance formula or Pythagorean Theorem***).

Example A

Write a two-column proof.



Given: $\overline{AB} \parallel \overline{DC}$ and $\overline{AB} \cong \overline{DC}$

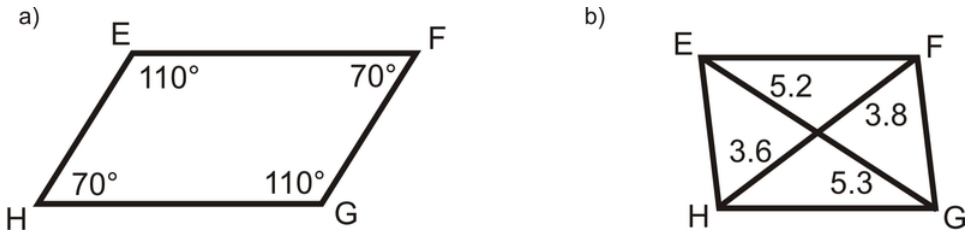
Prove: $ABCD$ is a parallelogram

TABLE 2.5:

Statement	***Reason***
1. $\overline{AB} \parallel \overline{DC}$ and $\overline{AB} \cong \overline{DC}$	Given
2. $\angle ABD \cong \angle BDC$	Alternate Interior Angles
3. $\overline{DB} \cong \overline{DB}$	Reflexive PoC
4. $\triangle ABD \cong \triangle CDB$	SAS
5. $\overline{AD} \cong \overline{BC}$	CPCTC
6. $ABCD$ is a parallelogram	Opposite Sides Converse

Example B

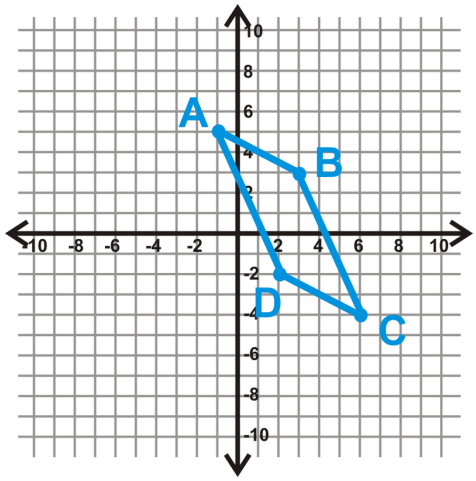
Is quadrilateral $EFGH$ a parallelogram? How do you know?



For part a, the opposite angles are equal, so by the Opposite Angles Theorem Converse, $EFGH$ is a parallelogram. In part b, the diagonals do not bisect each other, so $EFGH$ is not a parallelogram.

Example C

Is the quadrilateral $ABCD$ a parallelogram? What is the perimeter of $ABCD$?



First, find the length of AB and CD.

$$\begin{aligned} AB &= \sqrt{(-1-3)^2+(5-3)^2} \\ &= \sqrt{(-4)^2+2^2} \\ &= \sqrt{16+4} \\ &= \sqrt{20} \end{aligned} \qquad \begin{aligned} CD &= \sqrt{(2-6)^2+(-2+4)^2} \\ &= \sqrt{(-4)^2+2^2} \\ &= \sqrt{16+4} \\ &= \sqrt{20} \end{aligned}$$

$AB = CD$, so if the two lines have the same slope, $ABCD$ is a parallelogram.

Slope $AB = \frac{5-3}{-1-3} = \frac{2}{-4} = -\frac{1}{2}$ Slope $CD = \frac{-2+4}{2-6} = \frac{2}{-4} = -\frac{1}{2}$

Therefore, $ABCD$ is a parallelogram.

To find the perimeter, we need to find the lengths of all of the sides and add those sides together. We already know that $AB = CD = \sqrt{20}$. We also know, by the Opposite Sides Theorem, that $AD = BC$. If we find the length of one of the segments, we will have enough information to find the perimeter.

By creating a right triangle with an additional point at (6, 3) we can use the Pythagorean Theorem to find the length of BC as an alternative to using the distance formula.

$BC^2 = 3^2 + 7^2$
$BC^2 = 9 + 49$
$BC^2 = 58$
$BC = \sqrt{58}$

Since $AD = BC = \sqrt{58}$. Now that we have the lengths of all four sides, we can add them to find the perimeter.

$\sqrt{20} + \sqrt{20} + \sqrt{58} + \sqrt{58} = 2\sqrt{20} + 2\sqrt{58}$ units

If you choose to simplify your radicals, $4\sqrt{5} + 2\sqrt{58}$ units would be an equivalent expression.

Watch this video for help with the Examples above.

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CK-12 Foundation: Chapter6QuadrilateralsthatareParallelogramsB

Concept Problem Revisited

First, we can use the Pythagorean Theorem to find the length of the second diagonal.

$$\begin{aligned} 90^2 + 90^2 &= d^2 \\ 8100 + 8100 &= d^2 \\ 16200 &= d^2 \\ d &= 127.3 \end{aligned}$$

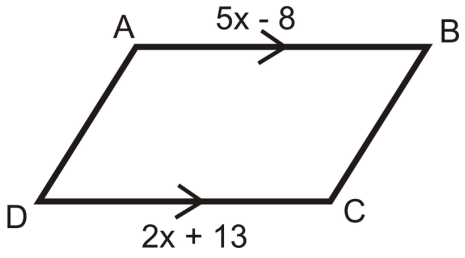
This means that the diagonals are equal. If the diagonals are equal, the other two sides of the diamond are also 90 feet. Therefore, the baseball diamond is a parallelogram.

Vocabulary

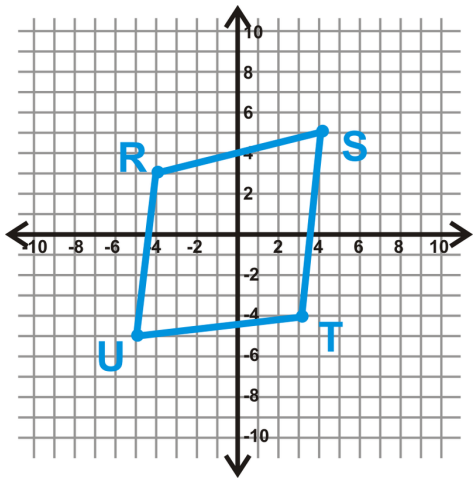
A ***parallelogram*** is a quadrilateral with two pairs of parallel sides.

Guided Practice

1. What value of x would make $ABCD$ a parallelogram?



2. Is the quadrilateral $RSTU$ a parallelogram?



3. If a quadrilateral has one pair of parallel sides is it a parallelogram?

Answers:

1. $\overline{AB} \parallel \overline{DC}$ from the markings. Therefore, $ABCD$ would be a parallelogram if $AB = DC$ as well.

$$5x - 8 = 2x + 13$$
$$3x = 21$$
$$x = 7$$

In order for $ABCD$ to be a parallelogram, x must equal 7.

2. Let's use the Parallelogram Diagonals Converse to determine if $RSTU$ is a parallelogram. Find the midpoint of each diagonal.

Midpoint of $RT = \left(\frac{-4+3}{2}, \frac{3-4}{2}\right) = (-0.5, -0.5)$

Midpoint of $SU = \left(\frac{4-5}{2}, \frac{5-5}{2}\right) = (-0.5, 0)$

Because the midpoint is not the same, $RSTU$ is not a parallelogram.

3. Although it has one pair of parallel sides, this quadrilateral is not a parallelogram because its opposite sides are not necessarily congruent.

Interactive Practice

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Practice

For questions 1-11, determine if the quadrilaterals are parallelograms. If they are, write a reason.

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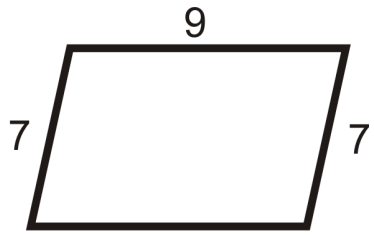

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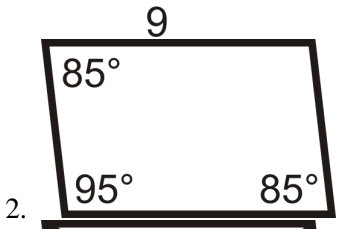

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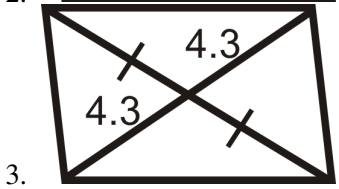

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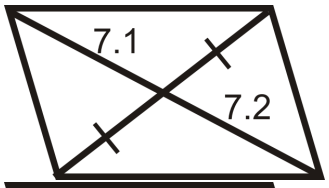

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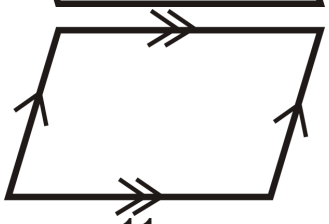

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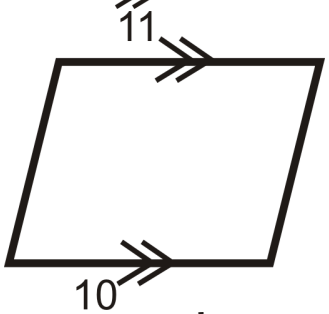

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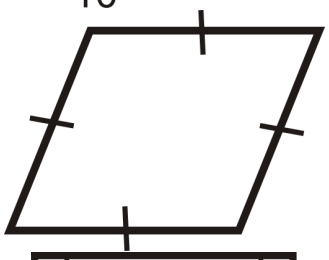

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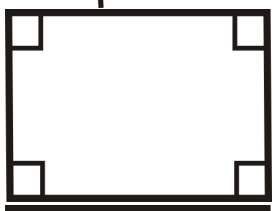

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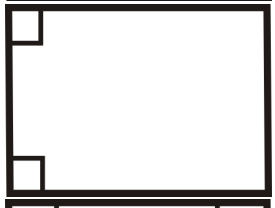

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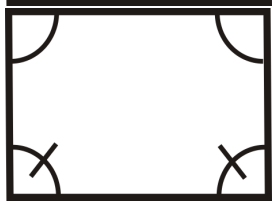
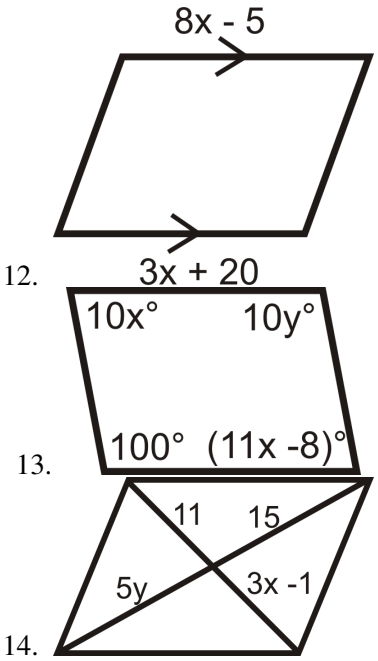


For questions 12-14, determine the value of x and y that would make the quadrilateral a parallelogram.

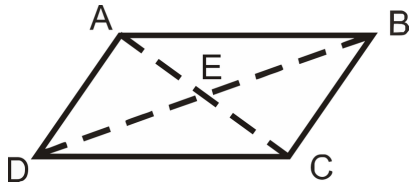


For questions 15-17, determine if $ABCD$ is a parallelogram.

15. $A(8,-1)$, $B(6,5)$, $C(-7,2)$, $D(-5,-4)$
16. $A(-5,8)$, $B(-2,9)$, $C(3,4)$, $D(0,3)$
17. $A(-2,6)$, $B(4,-4)$, $C(13,-7)$, $D(4,-10)$

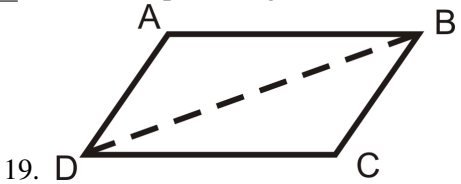
Write a two-column proof.

18. ***Parallelogram Diagonals Theorem Converse***



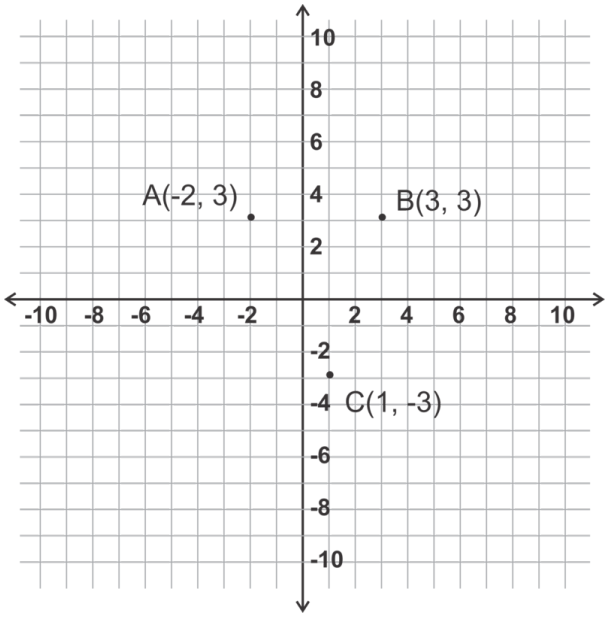
Given: $\overline{AE} \cong \overline{EC}$, $\overline{DE} \cong \overline{EB}$Prove: $ABCD$ is a parallelogram

19.



Given: $\angle ADB \cong CBD$, $\overline{AD} \cong \overline{BC}$Prove: $ABCD$ is a parallelogram

Suppose that $A(-2,3)$, $B(3,3)$ and $C(1,-3)$ are three of four vertices of a parallelogram.



20. Depending on where you choose to put point D, the name of the parallelogram you draw will change. Sketch a picture to show all possible parallelograms. How many can you draw?
21. If you know the parallelogram is named $ABDC$, what is the slope of side parallel to $\overline{AC}$?
22. Again, assuming the parallelogram is named $ABDC$, what is the length of $\overline{BD}$?

The points $Q(-1,1)$, $U(7,1)$, $A(1,7)$ and $D(-1,5)$ are the vertices of quadrilateral $QUAD$. Plot the points on graph paper to complete problems 23-26.

23. Find the midpoints of sides $\overline{QU}$, $\overline{UA}$, $\overline{AD}$ and $\overline{DQ}$. Label them W, X, Y and Z respectively.
24. Connect the midpoints to form quadrilateral $WXYZ$. What does this quadrilateral appear to be?
25. Find the perimeter of the polygon.
26. Use slopes to verify your answer to problem 24.
27. Use midpoints to verify your answer to problem 24.

2.3 Quadrilaterals

Here you will review different kinds of quadrilaterals and their properties.

All squares are rectangles, but not all rectangles are squares. How is this possible?

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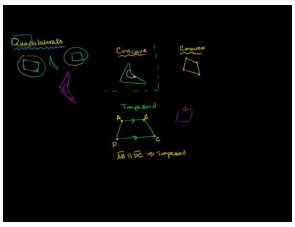

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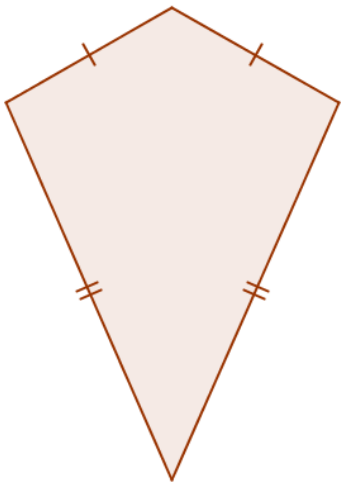
http://www.youtube.com/watch?v=Am9tMiUQatQ Khan Academy: Quadrilateral Overview

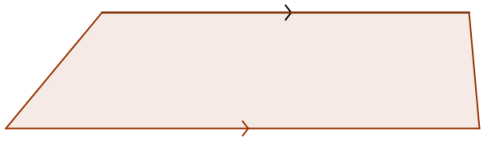
Guidance

A **quadrilateral** is a polygon with four sides. There are many common **special quadrilaterals** that you should be familiar with. Below, these special quadrilaterals are described with their definitions and some properties. Additional properties are explored further and proved in another concept.

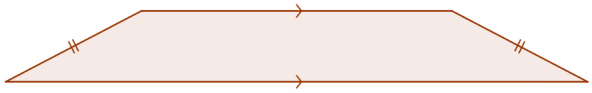
- A **kite** is a convex quadrilateral with two pairs of adjacent congruent sides such that not all sides are congruent. The word adjacent means "next to", so the congruent sides are next to each other. *Note: some texts leave out the stipulation that "not all sides are congruent". If this is the case, it is possible for a kite to be a rhombus if all four sides are congruent.* To note that sides are congruent on a picture, draw corresponding tick marks through the middles of the congruent sides.



- A **trapezoid** is a quadrilateral with *at least* one pair of parallel sides. *Note: most texts use the word "exactly", which means quadrilaterals with exactly one pair of parallel sides as pictured below. Since a trapezoid is a quadrilateral with at least one pair of parallel sides, all parallelograms are trapezoids*. To note that sides are parallel on a picture, draw corresponding arrow markings through the middles of the parallel sides.



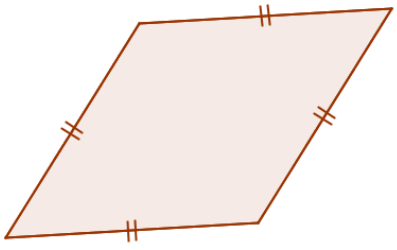
- An **isosceles trapezoid** is a trapezoid with one pair of congruent sides. *An additional property of isosceles trapezoids is base angles are congruent.*



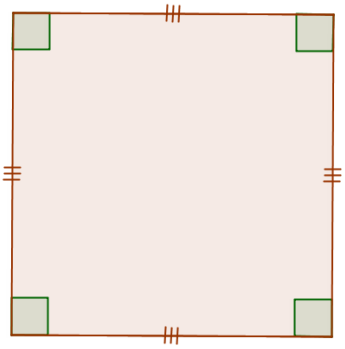
- A **parallelogram** is a quadrilateral with two pairs of parallel sides. *Some additional properties of parallelograms are opposite sides are congruent and opposite angles are congruent.*

- A **rectangle** is a quadrilateral with four right angles. All rectangles are parallelograms. *An additional property of rectangles is diagonals are congruent. You will be proving this in the practice section.*

- A **rhombus** is a quadrilateral with four congruent sides. All rhombuses are parallelograms.



- A **square** is a quadrilateral with four right angles and four congruent sides. All squares are rectangles and rhombuses.



Notice that these categories of quadrilaterals overlap. A square is not only a square, but also a rhombus, a rectangle, a parallelogram, a trapezoid, and a quadrilateral. This means that a square will have all the same properties as rhombuses, rectangles, parallelograms, trapezoids, and quadrilaterals.

The following figure shows the hierarchy of quadrilaterals.

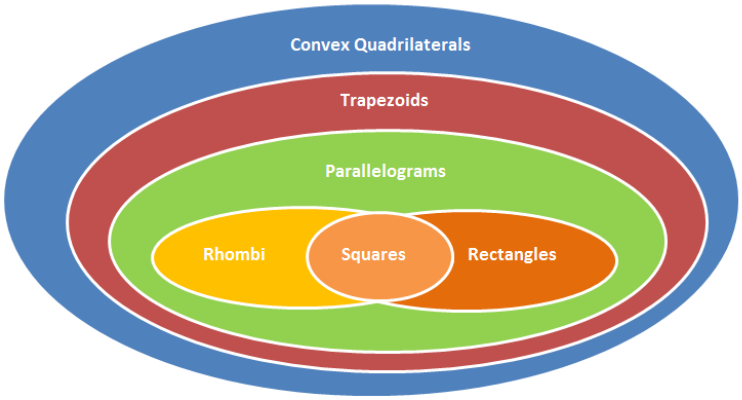


FIGURE 2.1

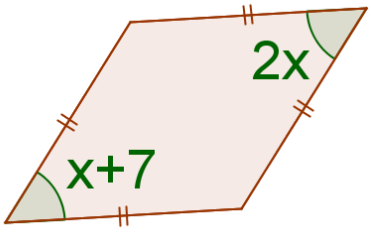
Example A

What types of quadrilaterals have four right angles?

Solution: Rectangles and squares have four right angles.

Example B

Solve for x (picture not drawn to scale).



Solution: This quadrilateral is marked as having four congruent sides, so it is a rhombus. Rhombuses are parallelograms, so they have all the same properties as parallelograms. One property of parallelograms is that opposite angles are congruent. This means that the marked angles in this rhombus must be congruent.

$$x+7=2x$$
$$x=7$$

Example C

A quadrilateral has two pairs of parallel sides. What type of quadrilateral **must** it be? What type of quadrilateral **could** it be?

Solution: Two pairs of parallel sides is the definition of a parallelogram, so this quadrilateral **must** be a parallelogram. It **could** be a rectangle, a square, a trapezoid, or a rhombus if it also satisfied the definition of any of those quadrilaterals.

Concept Problem Revisited

Rectangles are defined as quadrilaterals with four right angles. Squares are defined as quadrilaterals with four right angles and four congruent sides. Because all squares have four right angles and satisfy the definition for rectangles, they can all also be called rectangles. On the other hand, not all rectangles have four congruent sides, so not all rectangles can also be called squares.

Vocabulary

A ***kite*** is a quadrilateral with two pairs of adjacent congruent sides such that not all four sides are congruent. *Note: some texts leave out the stipulation that "not all sides are congruent". If this is the case, it is possible for a kite to be a rhombus if all four sides are congruent.*

A ***trapezoid*** is a quadrilateral with *at least* one pair of parallel sides.

An ***isosceles trapezoid*** is a trapezoid with one pair of congruent sides.

A ***parallelogram*** is a quadrilateral with two pairs of parallel sides.

A ***rectangle*** is a quadrilateral with four right angles.

A ***rhombus*** is a quadrilateral with four congruent sides.

A ***square*** is a quadrilateral with four right angles and four congruent sides.

Guided Practice

1. Draw a square. Draw in the diagonals of the square. Make at least one conjecture about the diagonals of the square.

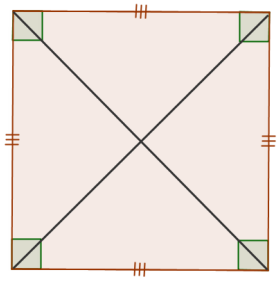
2. A quadrilateral has four congruent sides. What type of quadrilateral **must** it be? What type of quadrilateral **could** it be?

3. Solve for x (picture not drawn to scale).

Answers:

1. To make a conjecture means to make an educated guess. There are a few conjectures you might make about the diagonals of a square. These conjectures will be proved in a later concept. Here are some possible conjectures:

- diagonals of a square are congruent
- diagonals of a square are perpendicular
- diagonals of a square bisect each other (cut each other in half)
- diagonals of a square bisect the angles (cut the 90° angles in half)



2. It must be a rhombus and therefore also a parallelogram. It could be a square.

3. This is a parallelogram so opposite sides are congruent.

$$3x+1=5x-12$$
$$2x=13$$
$$x=6.5$$

Practice

Decide whether each statement is always, sometimes, or never true. Explain your answer.

1. A square is a rectangle.
2. A rhombus is a square.
3. An isosceles trapezoid is a trapezoid.
4. A parallelogram is a quadrilateral.
5. A square is a parallelogram.
6. A trapezoid is a parallelogram.

Decide what type of quadrilateral it **must** be and what type of quadrilateral it **could** be based on the description.

7. A quadrilateral has 4 congruent angles.

8. A quadrilateral has 2 pairs of congruent sides.

9. Draw a kite. Draw in its diagonals. Make at least one conjecture about the diagonals of kites.

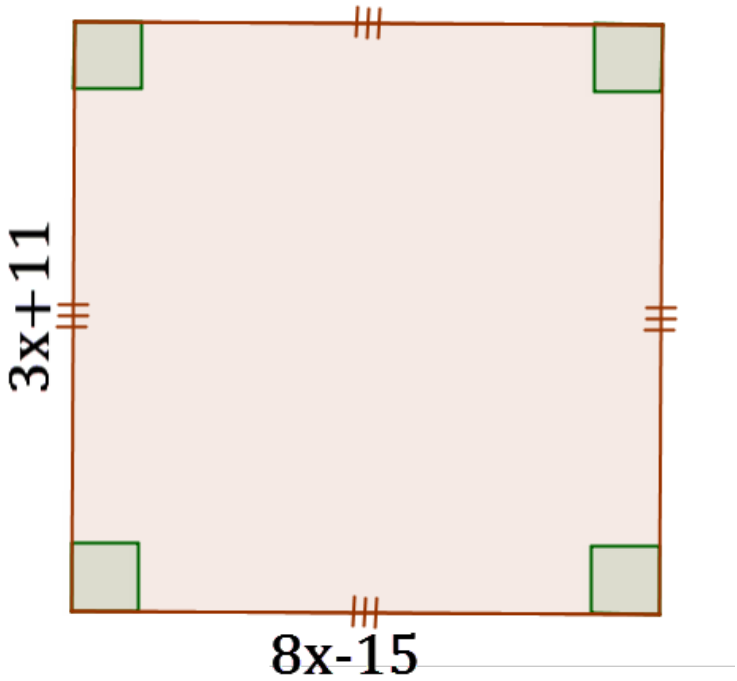
10. Draw a rectangle. Draw in its diagonals. Make at least one conjecture about the diagonals of rectangles.

11. Draw a rhombus. Draw in its diagonals. Make at least one conjecture about the diagonals of rhombuses.

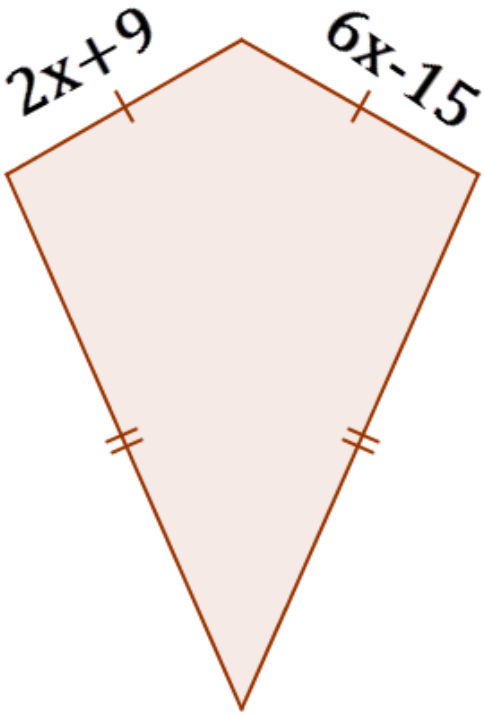
12. Draw a kite. Make a conjecture about the opposite angles of kites.

Use the markings on the shapes below to identify the shape. Then, solve for *x*. *Note: pictures are not drawn to scale.*

13.

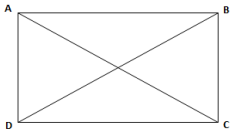


14.



15. Make a conjecture about the adjacent angles of a parallelogram (such as the ones marked in the picture below). How must they be related?

16. Given rectangle *ABCD*, prove the diagonals are congruent.



2.4 Quadrilateral Classification

Here you'll learn how to differentiate among parallelograms, rectangles, rhombuses, squares, kites, trapezoids, and quadrilaterals in the coordinate plane.

What if you were given the coordinates of four points that form a quadrilateral? How could you determine if that quadrilateral qualifies as one of the special four-sided figures you learned about in the previous Concepts? After completing this Concept, you'll be able to make such a determination.

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CK-12 Classifying Quadrilaterals in the Coordinate Plane

Guidance

In order to be successful in this concept you need to already be familiar with the definitions and properties of the following quadrilaterals: parallelograms, rhombuses, rectangles, squares, kites and trapezoids. The definitions for each are provided in the vocabulary section as a resource, and further information can be found by searching on those topic words.

When working in the coordinate plane, you will sometimes want to know what type of shape a given shape is. You should easily be able to tell that it is a quadrilateral if it has four sides. But how can you classify it beyond that?

First you should graph the shape if it has not already been graphed. Look at it and see if it looks like any special quadrilateral. Do the sides appear to be congruent? Do they meet at right angles? This will give you a place to start.

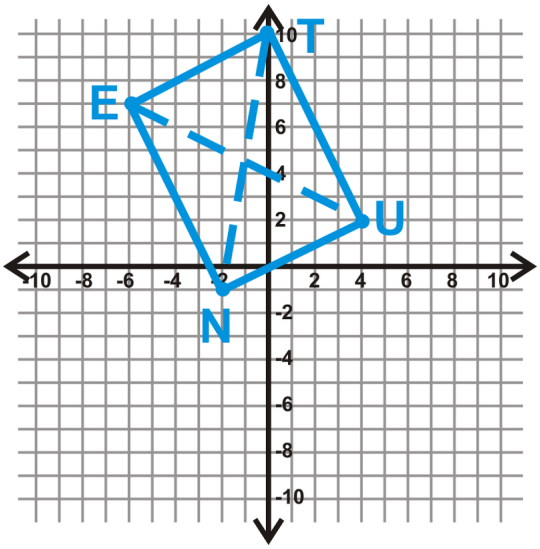
Once you have a guess for what type of quadrilateral it is, your job is to prove your guess. To prove that a quadrilateral is a parallelogram, rectangle, rhombus, square, kite or trapezoid, you must show that it meets the definition of that shape OR that it has properties that only that shape has.

If it turns out that your guess was wrong because the shape does not fulfill the necessary properties, you can guess again. If it appears to be no type of special quadrilateral then it is simply a *quadrilateral*.

The examples below will help you to see what this process might look like.

Example A

Determine what type of parallelogram $TUNE$ is: $T(0,10), U(4,2), N(-2,-1)$, and $E(-6,7)$.



This *looks* like a rectangle. Let's see if the diagonals are equal. If they are, then $TUNE$ is a rectangle.

$$\begin{aligned} EU &= \sqrt{(-6-4)^2+(7-2)^2} \\ &= \sqrt{(-10)^2+5^2} \\ &= \sqrt{100+25} \\ &= \sqrt{125} \end{aligned} \qquad \begin{aligned} TN &= \sqrt{(0+2)^2+(10+1)^2} \\ &= \sqrt{2^2+11^2} \\ &= \sqrt{4+121} \\ &= \sqrt{125} \end{aligned}$$

If the diagonals are also perpendicular, then $TUNE$ is a square.

Slope of $EU = \frac{7-2}{-6-4} = -\frac{5}{10} = -\frac{1}{2}$ Slope of $TN = \frac{10-(-1)}{0-(-2)} = \frac{11}{2}$

The slope of $EU \neq$ slope of TN, so $TUNE$ is a rectangle.

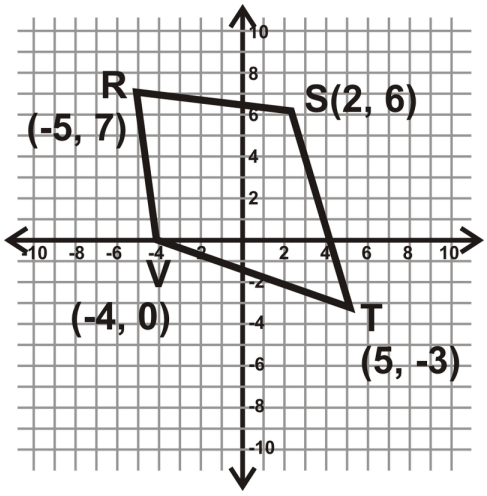
Example B

A quadrilateral is defined by the four lines $y = 2x+1$, $y = -x+5$, $y = 2x-4$, and $y = -x-5$. Is this quadrilateral a parallelogram?

To check if its a parallelogram we have to check that it has two pairs of parallel sides. From the equations we can see that the slopes of the lines are 2, -1, 2 and -1. Because two pairs of slopes match, this shape has two pairs of parallel sides and is a parallelogram.

Example C

Determine what type of quadrilateral $RSTV$ is.



This *looks* like a kite. Find the lengths of all the sides to check if the adjacent sides are congruent.

$$\begin{aligned} RS &= \sqrt{(-5-2)^2+(7-6)^2} \\ &= \sqrt{(-7)^2+1^2} \\ &= \sqrt{50} \end{aligned} \qquad \begin{aligned} ST &= \sqrt{(2-5)^2+(6-(-3))^2} \\ &= \sqrt{(-3)^2+9^2} \\ &= \sqrt{90} \end{aligned}$$

$$\begin{aligned} RV &= \sqrt{(-5-(-4))^2+(7-0)^2} \\ &= \sqrt{(-1)^2+7^2} \\ &= \sqrt{50} \end{aligned} \qquad \begin{aligned} VT &= \sqrt{(-4-5)^2+(0-(-3))^2} \\ &= \sqrt{(-9)^2+3^2} \\ &= \sqrt{90} \end{aligned}$$

From this we see that the adjacent sides are congruent. Therefore, $RSTV$ is a kite.

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CK-12 Classifying Quadrilaterals in the Coordinate Plane

Guided Practice

1. A quadrilateral is defined by the four lines $y = 2x+1$, $y = -2x+5$, $y = 2x-4$, and $y = -2x-5$. Is this quadrilateral a rectangle?

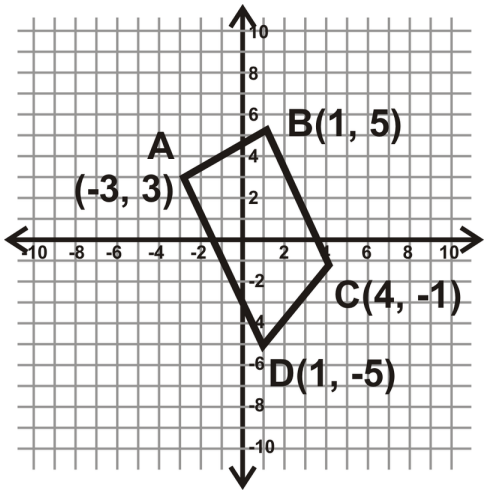
2. Determine what type of quadrilateral $ABCD$ is. $A(-3,3), B(1,5), C(4,-1), D(1,-5)$.

3. Determine what type of quadrilateral $EFGH$ is. $E(5,-1), F(11,-3), G(5,-5), H(-1,-3)$

Answers:

1. To be a rectangle a shape must have four right angles. This means that the sides must be perpendicular to each other. From the given equations we see that the slopes are 2, -2, 2 and -2. Because the slopes are not opposite reciprocals of each other, the sides are not perpendicular, and the shape is not a rectangle.

2. First, graph *ABCD*. This will make it easier to figure out what type of quadrilateral it is. From the graph, we can tell this is not a parallelogram. Find the slopes of $\overline{BC}$ and $\overline{AD}$ to see if they are parallel.



Slope of $\overline{BC} = \frac{5-(-1)}{1-4} = \frac{6}{-3} = -2$

Slope of $\overline{AD} = \frac{3-(-5)}{-3-1} = \frac{8}{-4} = -2$

$\overline{BC} \| \overline{AD}$, so *ABCD* is a trapezoid. To determine if it is an isosceles trapezoid, find *AB* and *CD*.

$$AB = \sqrt{(-3-1)^2+(3-5)^2} = \sqrt{(-4)^2+(-2)^2} = \sqrt{20} = 2\sqrt{5}$$

$$ST = \sqrt{(4-1)^2+(-1-(-5))^2} = \sqrt{3^2+4^2} = \sqrt{25} = 5$$

$AB \neq CD$, therefore this is noy an isosceles trapezoid.

3. We will not graph this example. Let's find the length of all four sides.

$$EF = \sqrt{(5-11)^2+(-1-(-3))^2} = \sqrt{(-6)^2+2^2} = \sqrt{40}$$

$$\sqrt{FG} = \sqrt{(11-5)^2+(-3-(-5))^2} = \sqrt{6^2+2^2} = \sqrt{40}$$

$$GH = \sqrt{(5-(-1))^2+(-5-(-3))^2} = \sqrt{6^2+(-2)^2} = \sqrt{40}$$

$$HE = \sqrt{(-1-5)^2+(-3-(-1))^2} = \sqrt{(-6)^2+(-2)^2} = \sqrt{40}$$

All four sides are equal. This quadrilateral is either a ***rhombus*** or a ***square***. Let's find the length of the diagonals.

$$EG = \sqrt{(5-5)^2+(-1-(-5))^2} = \sqrt{0^2+4^2} = \sqrt{16} = 4$$

$$FH = \sqrt{(11-(-1))^2+(-3-(-3))^2} = \sqrt{12^2+0^2} = \sqrt{144} = 12$$

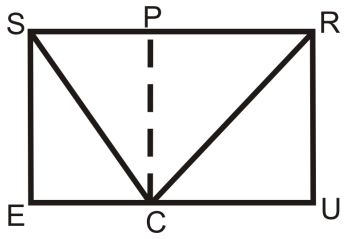
The diagonals are not congruent, so $EFGH$ is a rhombus and not a square.

Practice

Determine what type of quadrilateral $ABCD$ is.

1. $A(-2,4), B(-1,2), C(-3,1), D(-4,3)$
2. $A(-2,3), B(3,4), C(2,-1), D(-3,-2)$
3. $A(1,-1), B(7,1), C(8,-2), D(2,-4)$
4. $A(10,4), B(8,-2), C(2,2), D(4,8)$
5. $A(0,0), B(5,0), C(0,4), D(5,4)$
6. $A(-1,0), B(0,1), C(1,0), D(0,-1)$
7. $A(2,0), B(3,5), C(5,0), D(6,5)$

$SRUE$ is a rectangle and $PRUC$ is a square.



8. What type of quadrilateral is $SPCE$?
9. If $SR = 20$ and $RU = 12$, find CE.
10. Find SC and RC based on the information from part b. Round your answers to the nearest hundredth.

For questions 11-14, determine what type of quadrilateral $ABCD$ is. If it is no type of special quadrilateral, just write *quadrilateral*.

11. $A(1,-2), B(7,-5), C(4,-8), D(-2,-5)$
12. $A(6,6), B(10,8), C(12,4), D(8,2)$
13. $A(-1,8), B(1,4), C(-5,-4), D(-5,6)$
14. $A(5,-1), B(9,-4), C(6,-10), D(3,-5)$

2.5 References

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CHAPTER 3 Circles

Chapter Outline

3.1 SEGMENTS FROM CHORDS

3.2 ARCS IN CIRCLES

3.3 CHORDS IN CIRCLES

3.4 INSCRIBED ANGLES IN CIRCLES

3.5 INSCRIBED POLYGONS IN CIRCLES

3.6 SEGMENTS FROM SECANTS

3.7 SEGMENTS FROM SECANTS AND TANGENTS

3.8 ANGLES ON AND INSIDE A CIRCLE

3.9 CIRCLES SIMILARITY & ARC LENGTH

3.10 AREA OF SECTORS AND SEGMENTS

3.11 WRITING AND GRAPHING THE EQUATIONS OF CIRCLES

3.1 Segments from Chords

Here you'll learn the relationship that exists between two chords when they intersect to solve for unknown circle values.

What if you were given a circle with two chords that intersect each other? How could you use the length of some of the segments formed by their intersection to determine the lengths of the unknown segments? After completing this Concept, you'll be able to use the Intersecting Chords Theorem to solve problems like this one.

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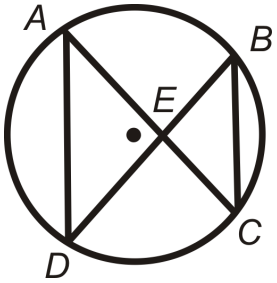
Segments from Chords CK-12

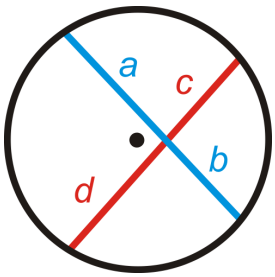
Guidance

When we have two chords that intersect inside a circle, as shown below, the two triangles that result are similar.



This makes the corresponding sides in each triangle proportional and leads to a relationship between the segments of the chords, as stated in the Intersecting Chords Theorem.

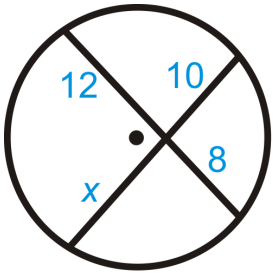
Intersecting Chords Theorem: If two chords intersect inside a circle so that one is divided into segments of length a and b and the other into segments of length c and d then $ab = cd$.



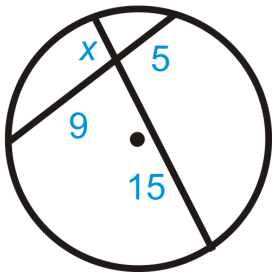
Example A

Find x in each diagram below.

a)



b)



Use the formula from the Intersecting Chords Theorem.

a)

$$12 \cdot 8 = 10 \cdot x$$
$$96 = 10x$$
$$9.6 = x$$

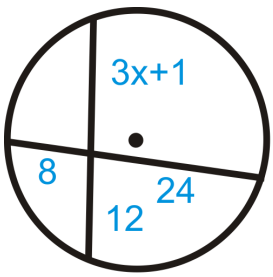
b)

$$x \cdot 15 = 5 \cdot 9$$
$$15x = 45$$
$$x = 3$$

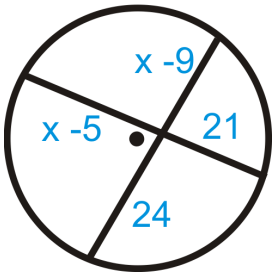
Example B

Solve for x.

a)



b)



Use the Intersecting Chords Theorem.

a)

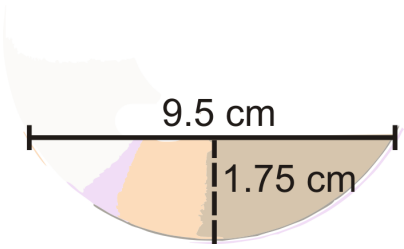
$$\begin{aligned} 8 \cdot 24 &= (3x+1) \cdot 12 \\ 192 &= 36x+12 \\ 180 &= 36x \\ 5 &= x \end{aligned}$$

b)

$$\begin{aligned} (x-5)21 &= (x-9)24 \\ 21x-105 &= 24x-216 \\ 111 &= 3x \\ 37 &= x \end{aligned}$$

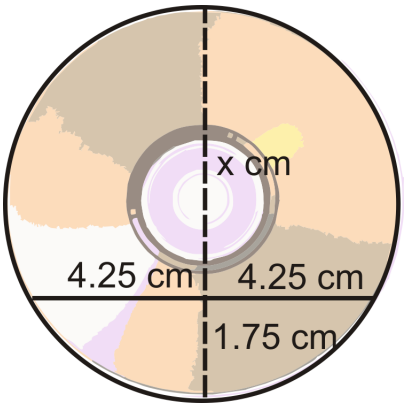
Example C

Ishmael found a broken piece of a CD in his car. He places a ruler across two points on the rim, and the length of the chord is 9.5 cm. The distance from the midpoint of this chord to the nearest point on the rim is 1.75 cm. Find the diameter of the CD.



Think of this as two chords intersecting each other. If we were to extend the 1.75 cm segment, it would be a diameter. So, if we find x in the diagram below and add it to 1.75 cm, we would find the diameter.

$$\begin{aligned} 4.25 \cdot 4.25 &= 1.75 \cdot x \\ 18.0625 &= 1.75x \\ x &\approx 10.3 \text{ } cm, \text{ making the diameter} 10.3+1.75 \approx 12 \text{ } cm, \text{ which is the actual diameter of a CD.} \end{aligned}$$



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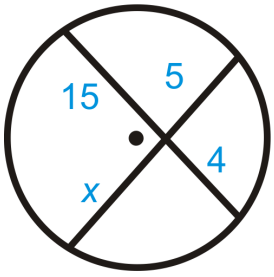
Segments from Chords CK-12

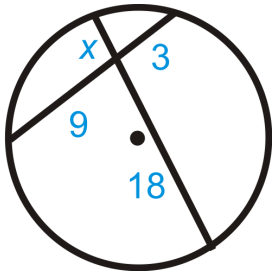
Guided Practice

Find x in each diagram below. Simplify any radicals.

1.



2.



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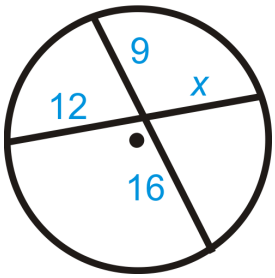


Answers

For all problems, use the Intersecting Chords Theorem.

1.

$$15 \cdot 4 = 5 \cdot x$$
$$60 = 5x$$
$$x = 12$$

2.

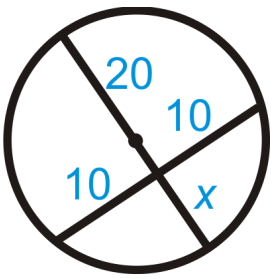
$$18 \cdot x = 9 \cdot 3$$
$$18x = 27$$
$$x = 1.5$$

3.

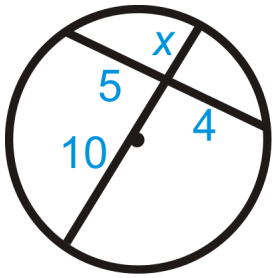
$$12 \cdot x = 9 \cdot 16$$
$$12x = 144$$
$$x = 12$$

Practice

Fill in the blanks for each problem below and then solve for the missing segment.

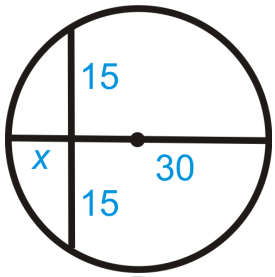
1. 

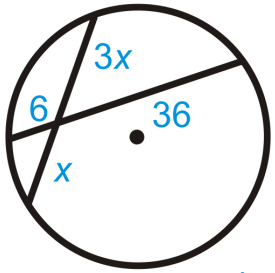
$20x =$ ___

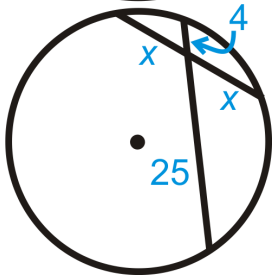
2. 

$__\cdot 4 = __\cdot x$

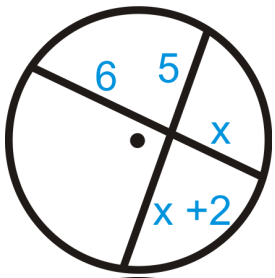
Find x in each diagram below. Simplify any radicals.

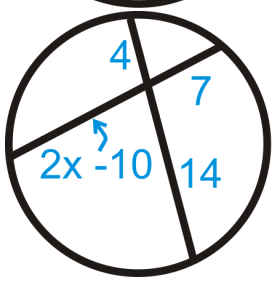
3. 

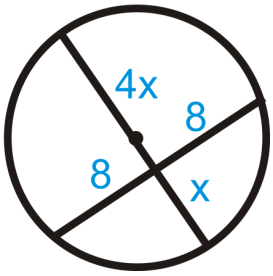
4. 

5. 

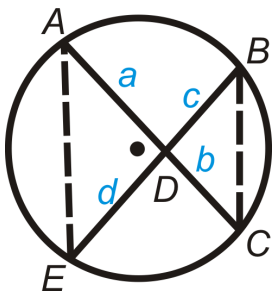
Find the value of x.

6. 

7. 

8. 

9. Suzie found a piece of a broken plate. She places a ruler across two points on the rim, and the length of the chord is 6 inches. The distance from the midpoint of this chord to the nearest point on the rim is 1 inch. Find the diameter of the plate.
10. Fill in the blanks of the proof of the Intersecting Chords Theorem.



Given: Intersecting chords $\overline{AC}$ and $\overline{BE}$.

Prove: $ab = cd$

TABLE 3.1:

Statement	***Reason***
1. Intersecting chords $\overline{AC}$ and $\overline{BE}$ with segments a, b, c, and d.	1.
2.	2. Congruent Inscribed Angles Theorem
3. $\triangle ADE \sim \triangle BDC$	3.
4.	4. Corresponding parts of similar triangles are proportional
5. $ab = cd$	5.

3.2 Arcs in Circles

Here you'll learn the properties of arcs and central angles of circles and how to apply them.

What if the Ferris wheel below had equally spaced seats, such that the central angle were 20°. How many seats are there? Why do you think it is important to have equally spaced seats on a Ferris wheel?

If the radius of this Ferris wheel is 25 ft., how far apart are two adjacent seats? Round your answer to the nearest tenth. *The shortest distance between two points is a straight line.* After completing this Concept, you'll be able to answer questions like these.

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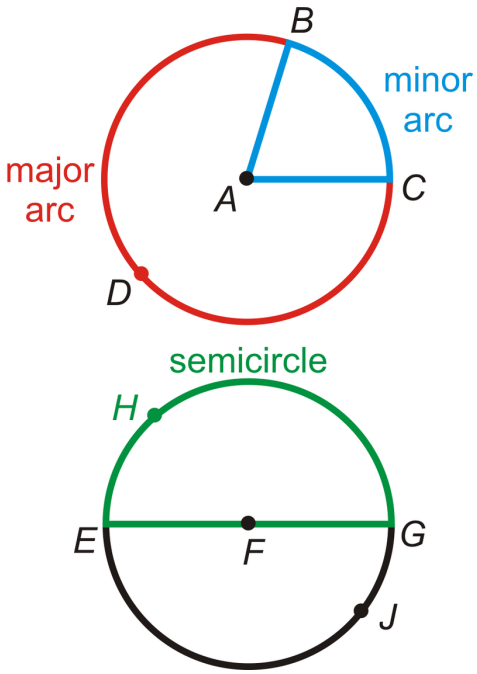
CK-12 Foundation: Chapter9ArcsinCirclesA

Learn more about central angles by watching the video at this link.

Guidance

A **central angle** is the angle formed by two radii of the circle with its vertex at the center of the circle. In the picture below, the central angle would be $\angle BAC$. Every central angle divides a circle into two **arcs** (an **arc** is a section of the circle). In this case the arcs are $\overset{\frown}{BC}$ and $\overset{\frown}{BDC}$. Notice the arc above the letters. To label an arc, always use this curve above the letters. Do not confuse $\overline{BC}$ and $\overset{\frown}{BC}$.



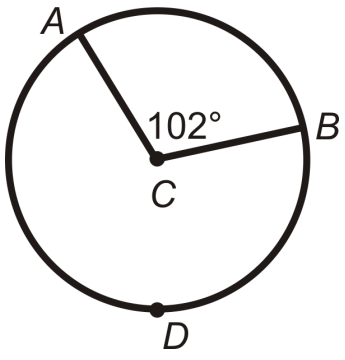
If D was not on the circle, we would not be able to tell the difference between $\widehat{BC}$ and $\widehat{BDC}$. There are 360° in a circle, where a semicircle is half of a circle, or 180°. $m\angle EFG = 180°$, because it is a straight angle, so $m\widehat{EHG} = 180°$ and $m\widehat{EJG} = 180°$.

- **Semicircle:** An arc that measures 180°.
- **Minor Arc:** An arc that is less than 180°.
- **Major Arc:** An arc that is greater than 180°. ***Always*** use 3 letters to label a major arc.

Two arcs are **congruent** if their central angles are congruent. The measure of the arc formed by two adjacent arcs is the sum of the measures of the two arcs (**Arc Addition Postulate**). An arc can be measured in degrees or in a linear measure (cm, ft, etc.). In this chapter we will use degree measure. ***The measure of the minor arc is the same as the measure of the central angle*** that corresponds to it. The measure of the major arc equals to 360° minus the measure of the minor arc. In order to prevent confusion, major arcs are always named with three letters; the letters that denote the endpoints of the arc and any other point on the major arc. When referring to the measure of an arc, always place an "m" in from of the label.

Example A

Find $m\widehat{AB}$ and $m\widehat{ADB}$ in $\odot C$.

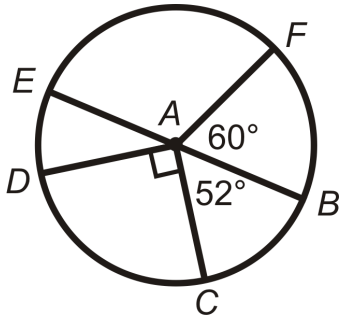


$m\widehat{AB} = m\angle ACB$. So, $m\widehat{AB} = 102^\circ$.

$$m\widehat{ADB} = 360^\circ - m\widehat{AB} = 360^\circ - 102^\circ = 258^\circ$$

Example B

Find the measures of the minor arcs in $\odot A$. $\overline{EB}$ is a diameter.

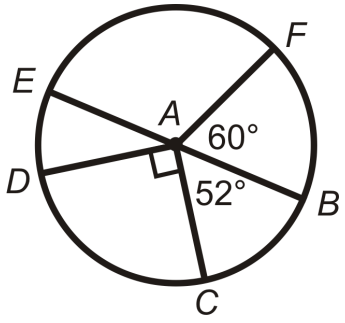


Because $\overline{EB}$ is a diameter, $m\angle EAB = 180^\circ$. Each arc has the same measure as its corresponding central angle.

$$\begin{aligned}
m\widehat{BF} &= m\angle FAB = 60^\circ \\
m\widehat{EF} &= m\angle EAF = 120^\circ \rightarrow 180^\circ - 60^\circ \\
m\widehat{ED} &= m\angle EAD = 38^\circ \ \rightarrow 180^\circ - 90^\circ - 52^\circ \\
m\widehat{DC} &= m\angle DAC = 90^\circ \\
m\widehat{BC} &= m\angle BAC = 52^\circ
\end{aligned}$$

Example C

Find the measures of the indicated arcs in $\odot A$. $\overline{EB}$ is a diameter.



a) $m\widehat{FED}$

b) $m\widehat{CDF}$

c) $m\widehat{DFC}$

Use the Arc Addition Postulate.

a) $m\widehat{FED} = m\widehat{FE} + m\widehat{ED} = 120^\circ + 38^\circ = 158^\circ$

b) $m\widehat{CDF} = m\widehat{CD} + m\widehat{DE} + m\widehat{EF} = 90^\circ + 38^\circ + 120^\circ = 248^\circ$

c) $m\widehat{DFC} = m\widehat{ED} + m\widehat{EF} + m\widehat{FB} + m\widehat{BC} = 38^\circ + 120^\circ + 60^\circ + 52^\circ = 270^\circ$

Watch this video for help with the Examples above.

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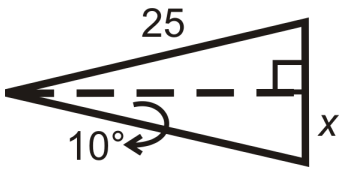
CK-12 Foundation: Chapter9ArcsinCirclesB

Concept Problem Revisited

Because the seats are 20° apart, there will be $\frac{360^\circ}{20^\circ} = 18$ seats. It is important to have the seats evenly spaced for balance. To determine how far apart the adjacent seats are, use the triangle to the right. We will need to use sine to find x and then multiply it by 2.



$$\sin 10^\circ = \frac{x}{25}$$
$$x = 25 \sin 10^\circ = 4.3 \ ft.$$

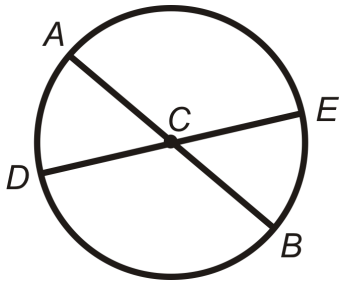
The total distance apart is 8.6 feet.

Vocabulary

A ***circle*** is the set of all points that are the same distance away from a specific point, called the ***center***. An ***arc*** is a section of the circle. A ***semicircle*** is an arc that measures 180°. A ***central angle*** is the angle formed by two radii with its vertex at the center of the circle. A ***minor arc*** is an arc that is less than 180°. A ***major arc*** is an arc that is greater than 180°.

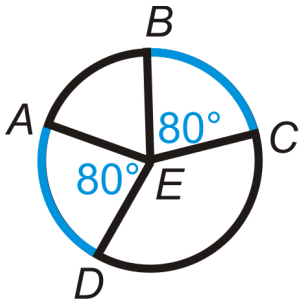
Guided Practice

1. List the congruent arcs in $\odot C$ below. $\overline{AB}$ and $\overline{DE}$ are diameters.

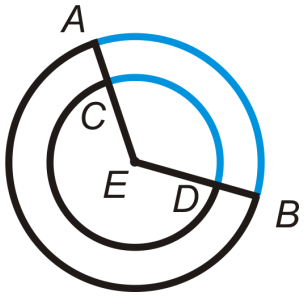


2. Are the blue arcs congruent? Explain why or why not.

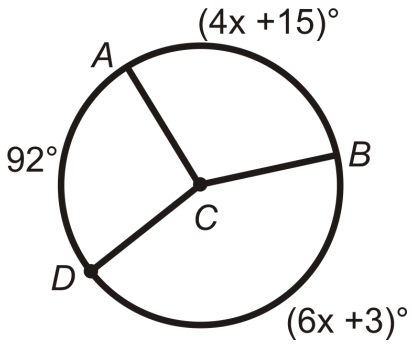
a)



b)



3. Find the value of x for $\odot C$ below.



Answers:

1. $\angle ACD \cong \angle ECB$ because they are vertical angles. $\angle DCB \cong \angle ACE$ because they are also vertical angles.
$\widehat{AD} \cong \widehat{EB}$ and $\widehat{AE} \cong \widehat{DB}$

2. a) $\widehat{AD} \cong \widehat{BC}$ because they have the same central angle measure and are in the same circle.

b) The two arcs have the same measure, but are not congruent because the circles have different radii.

3. The sum of the measure of the arcs is 360° because they make a full circle.

$$\begin{aligned} m\widehat{AB} + m\widehat{AD} + m\widehat{DB} &= 360^\circ \\ (4x+15)^\circ + 92^\circ + (6x+3)^\circ &= 360^\circ \\ 10x + 110^\circ &= 360^\circ \\ 10x &= 250 \\ x &= 25 \end{aligned}$$

Interactive Practice

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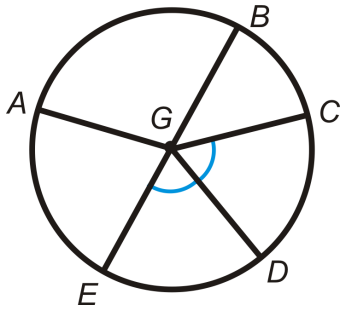
Practice

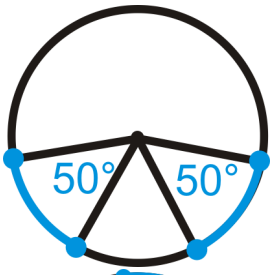
Determine if the arcs below are a minor arc, major arc, or semicircle of $\odot G$. $\overline{EB}$ is a diameter.



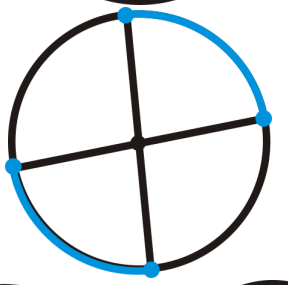
1. $\widehat{AB}$
2. $\widehat{ABD}$
3. $\widehat{BCE}$
4. $\widehat{CAE}$
5. $\widehat{ABC}$
6. $\widehat{EAB}$
7. Are there any congruent arcs? If so, list them.
8. If $m\widehat{BC} = 48^\circ$, find $m\widehat{CD}$.
9. Using #8, find $m\widehat{CAE}$.

Determine if the blue arcs are congruent. If so, state why.

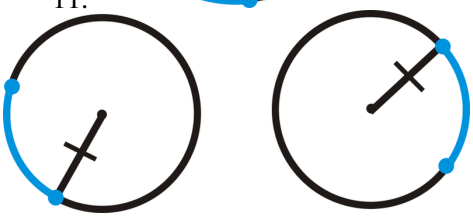
10.



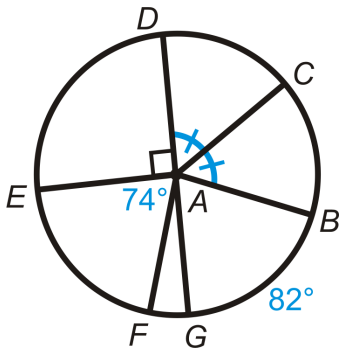
11.



12.

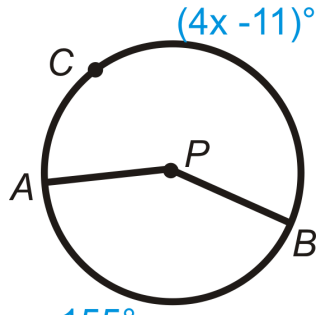


Find the measure of the indicated arcs or central angles in $\odot A$. $\overline{DG}$ is a diameter.

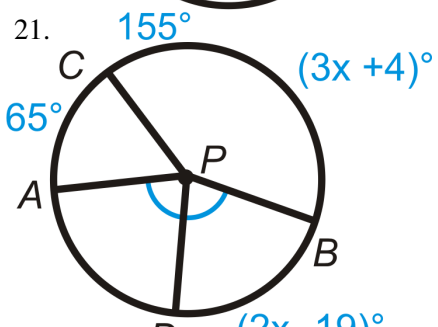


13. $\overset{\frown}{DE}$
14. $\overset{\frown}{DC}$
15. $\angle GAB$
16. $\overset{\frown}{FG}$
17. $\overset{\frown}{EDB}$
18. $\angle EAB$
19. $\overset{\frown}{DCF}$
20. $\overset{\frown}{DBE}$

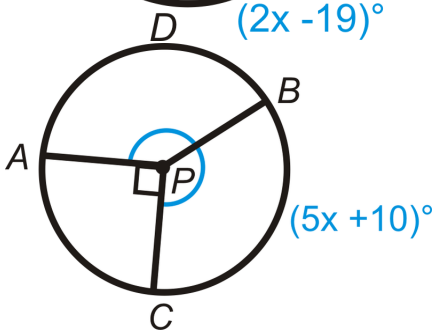
Algebra Connection Find the measure of x in $\odot P$.



21.

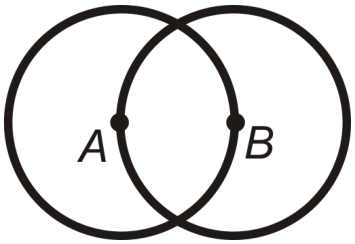


22.



23.

24. What can you conclude about $\odot A$ and $\odot B$?



3.3 Chords in Circles

Here you'll learn four theorems about chords in circles and how to apply them to find missing values.

What if you were given a circle with two chords drawn through it? How could you determine if these two chords were congruent? After completing this Concept, you'll be able to use four chord theorems to solve problems like this one.

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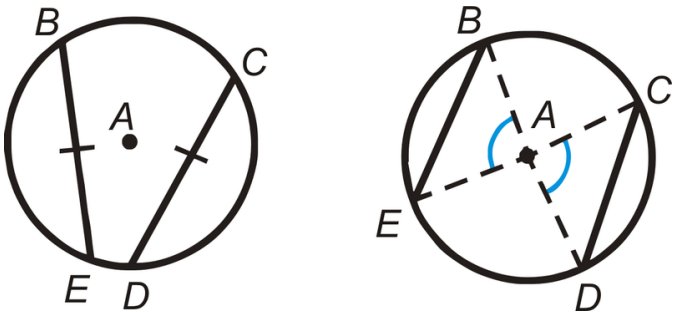
Chords in Circles CK-12

Guidance

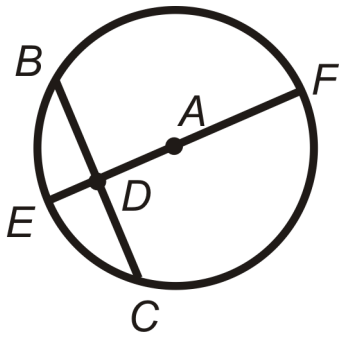
There are several important theorems about chords that will help you to analyze circles better.

1) **Chord Theorem #1:** In the same circle or congruent circles, minor arcs are congruent if and only if their corresponding chords are congruent.



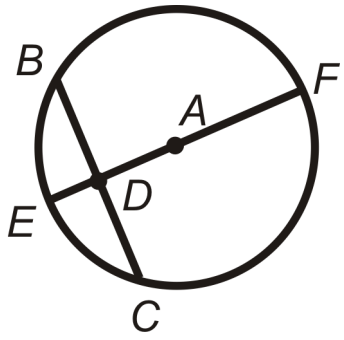
In both of these pictures, $\overline{BE} \cong \overline{CD}$ and $\widehat{BE} \cong \widehat{CD}$.

2) **Chord Theorem #2:** The perpendicular bisector of a chord is also a diameter.



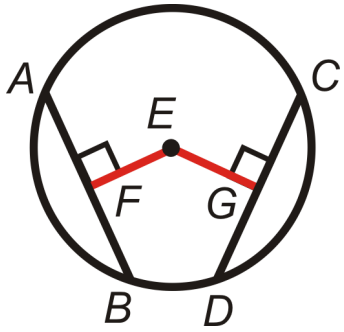
If $\overline{AD} \perp \overline{BC}$ and $\overline{BD} \cong \overline{DC}$ then $\overline{EF}$ is a diameter.

3) **Chord Theorem #3:** If a diameter is perpendicular to a chord, then the diameter bisects the chord and its corresponding arc.



If $\overline{EF} \perp \overline{BC}$, then $\overline{BD} \cong \overline{DC}$ and $\widehat{BE} \cong \widehat{EC}$.

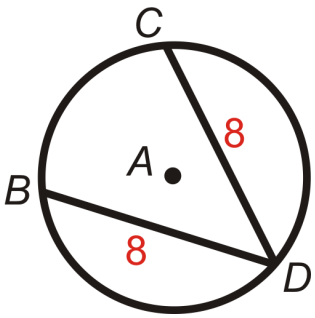
4) **Chord Theorem #4:** In the same circle or congruent circles, two chords are congruent if and only if they are equidistant from the center.



The shortest distance from any point to a line is the perpendicular line between them. If $FE = EG$ and $\overline{EF} \perp \overline{EG}$, then $\overline{AB}$ and $\overline{CD}$ are equidistant to the center and $\overline{AB} \cong \overline{CD}$.

Example A

Use $\odot A$ to answer the following.



a) If $m\widehat{BD} = 125°$, find $m\widehat{CD}$.

b) If $m\widehat{BC} = 80°$, find $m\widehat{CD}$.

Answers:

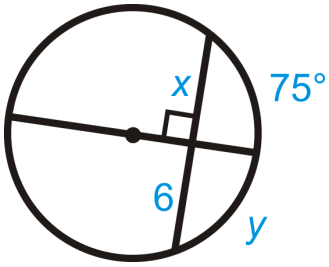
a) $BD = CD$, which means the arcs are congruent too. $m\widehat{CD} = 125°$.

b) $m\widehat{CD} \cong m\widehat{BD}$ because $BD = CD$.

$$\begin{aligned} m\widehat{BC} + m\widehat{CD} + m\widehat{BD} &= 360° \\ 80° + 2m\widehat{CD} &= 360° \\ 2m\widehat{CD} &= 280° \\ m\widehat{CD} &= 140° \end{aligned}$$

Example B

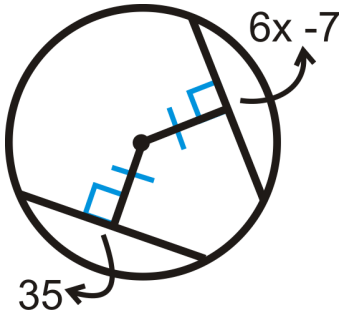
Find the values of x and y.



The diameter is perpendicular to the chord. From Chord Theorem #3, $x = 6$ and $y = 75°$.

Example C

Find the value of x.



Because the distance from the center to the chords is equal, the chords are congruent.

$$\begin{aligned} 6x - 7 &= 35 \\ 6x &= 42 \\ x &= 7 \end{aligned}$$

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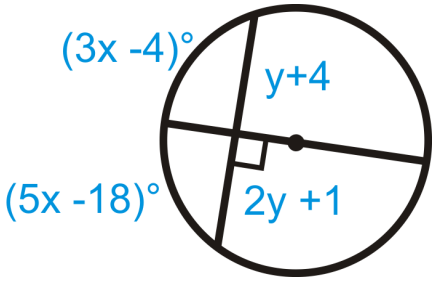
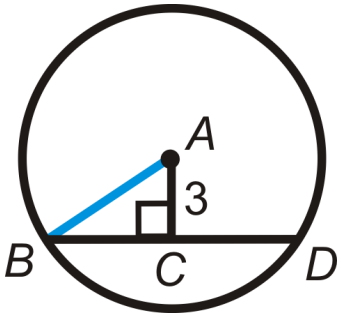
Chords in Circles CK-12

Guided Practice

1. Find the value of x and y.



2. $BD = 12$ and $AC = 3$ in $\odot A$. Find the radius.



3. Find $m\widehat{BD}$ from #2.

Answers:

1. The diameter is perpendicular to the chord, which means it bisects the chord and the arc. Set up equations for x and y.

$$(3x-4)^\circ = (5x-18)^\circ \qquad y+4 = 2y+1$$
$$14 = 2x \qquad 3 = y$$
$$7 = x$$

2. First find the radius. $\overline{AB}$ is a radius, so we can use the right triangle $\triangle ABC$ with hypotenuse $\overline{AB}$. From Chord Theorem #3, $BC = 6$.

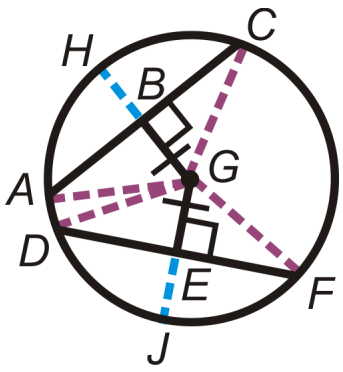
$$3^2 + 6^2 = AB^2$$
$$9 + 36 = AB^2$$
$$AB = \sqrt{45} = 3\sqrt{5}$$

3. First, find the corresponding central angle, $\angle BAD$. We can find $m\angle BAC$ using the tangent ratio. Then, multiply $m\angle BAC$ by 2 for $m\angle BAD$ and $m\overset{\frown}{BD}$.

$$\tan^{-1}\left(\frac{6}{3}\right) = m\angle BAC$$
$$m\angle BAC \approx 63.43^\circ$$
$$m\angle BAD \approx 2 \cdot 63.43^\circ \approx 126.86^\circ \approx m\overset{\frown}{BD}$$

Practice

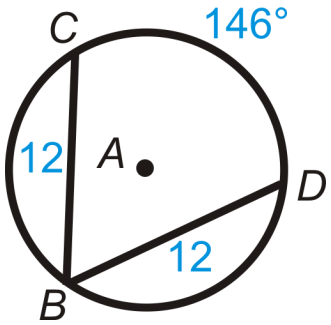
Fill in the blanks.



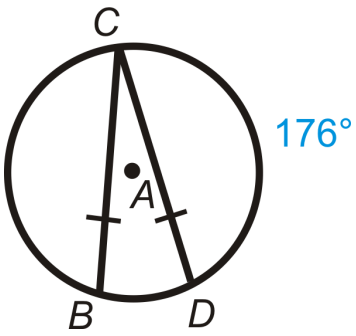
1. $____ \cong \overline{DF}$
2. $\overset{\frown}{AC} \cong ____$
3. $\overset{\frown}{DJ} \cong ____$
4. $____ \cong \overline{EJ}$
5. $\angle AGH \cong ____$
6. $\angle DGF \cong ____$
7. List all the congruent radii in $\odot G$.

Find the value of the indicated arc in $\odot A$.

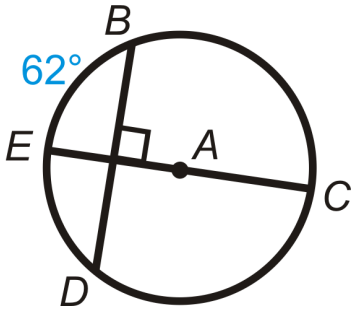
8. $m\overset{\frown}{BC}$



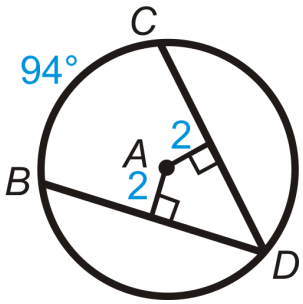
9. $m\overset{\frown}{BD}$



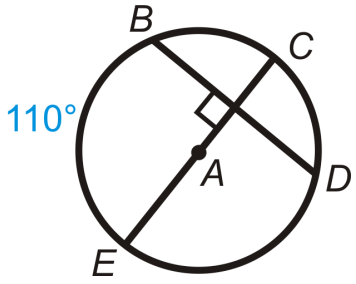
10. $m\widehat{BC}$



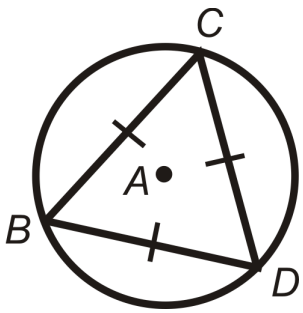
11. $m\widehat{BD}$



12. $m\widehat{BD}$

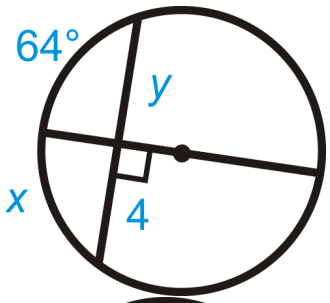


13. $m\widehat{BD}$



Find the value of x and/or y.

14.



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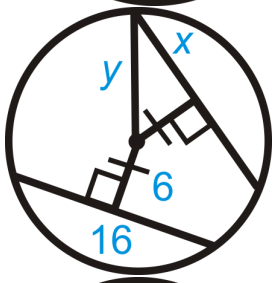

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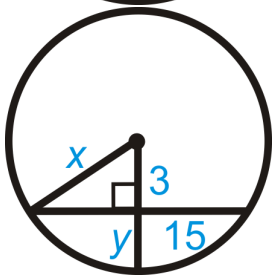
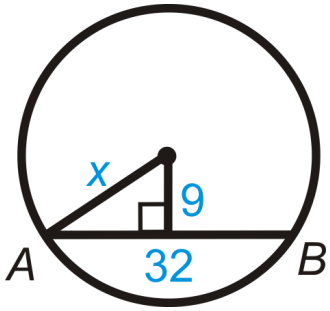


17. $AB = 32$



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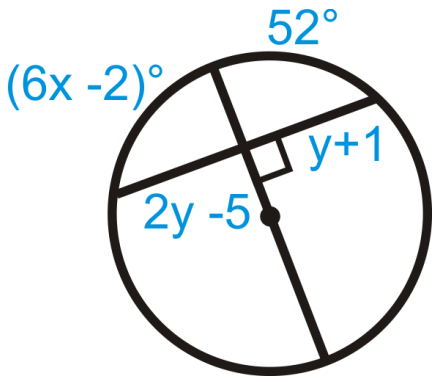

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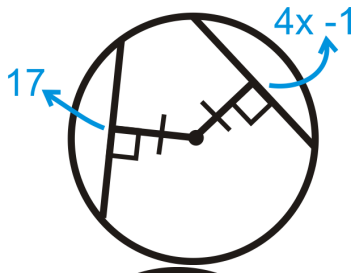

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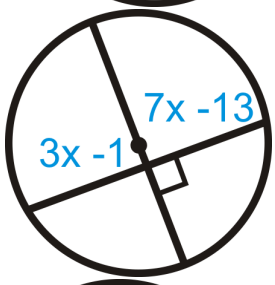
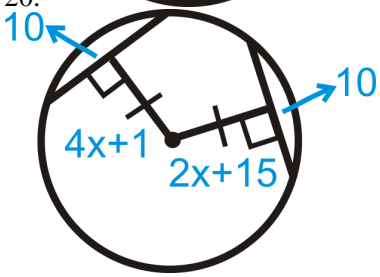
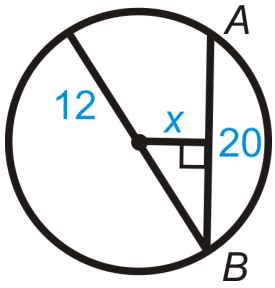


21.



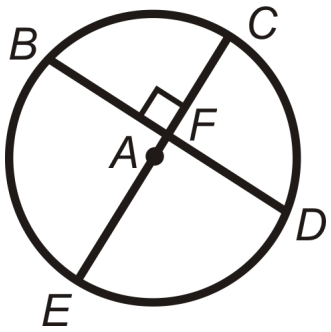
22. $AB = 20$



23. Find $m\widehat{AB}$ in Question 17. Round your answer to the nearest tenth of a degree.
24. Find $m\widehat{AB}$ in Question 22. Round your answer to the nearest tenth of a degree.

In problems 25-27, what can you conclude about the picture? State a theorem that justifies your answer. You may assume that A is the center of the circle.

25.



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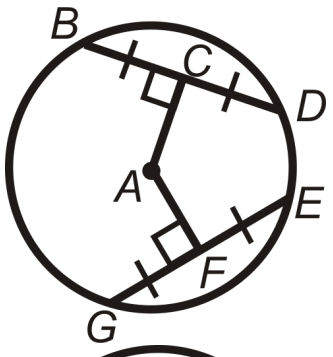

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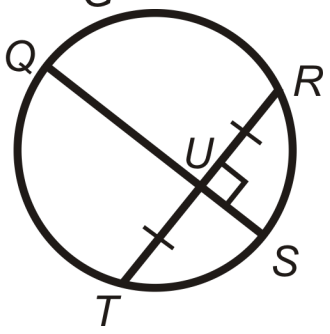

3.4 Inscribed Angles in Circles

Here you'll learn the Inscribed Angle Theorem, which states that the measure of an inscribed angle is half the measure of its intercepted arc. You'll also learn other inscribed angle theorems and you'll use them to solve problems about circles.

What if you had a circle with two chords that share a common endpoint? How could you use the arc formed by those chords to determine the measure of the angle those chords make inside the circle? After completing this Concept, you'll be able to use the Inscribed Angle Theorem to solve problems like this one.

Watch This

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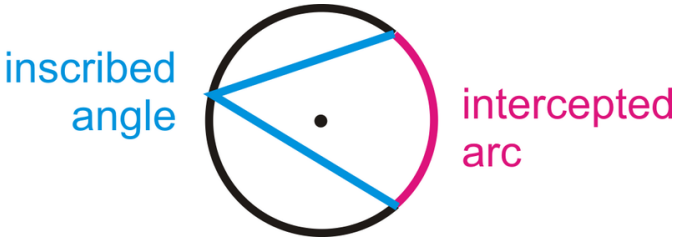
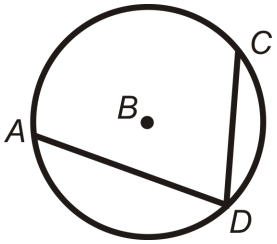
Inscribed Angles in Circles CK-12

Guidance

An **inscribed angle** is an angle with its vertex on the circle and whose sides are chords. The **intercepted arc** is the arc that is inside the inscribed angle and whose endpoints are on the angle. The vertex of an inscribed angle can be anywhere on the circle as long as its sides intersect the circle to form an intercepted arc.

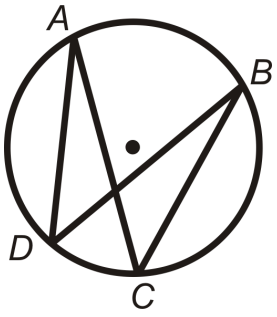


The **Inscribed Angle Theorem** states that the measure of an inscribed angle is half the measure of its intercepted arc.



$m\angle ADC = \frac{1}{2}m\widehat{AC}$ and $m\widehat{AC} = 2m\angle ADC$

Inscribed angles that intercept the same arc are congruent. This is called the **Congruent Inscribed Angles Theorem** and is shown below.



$\angle ADB$ and $\angle ACB$ intercept $\overset{\frown}{AB}$, so $m\angle ADB = m\angle ACB$. Similarly, $\angle DAC$ and $\angle DBC$ intercept $\overset{\frown}{DC}$, so $m\angle DAC = m\angle DBC$.

An angle intercepts a semicircle if and only if it is a right angle (**Semicircle Theorem**). ***Anytime a right angle is inscribed in a circle, the endpoints of the angle are the endpoints of a diameter and the diameter is the hypotenuse.***

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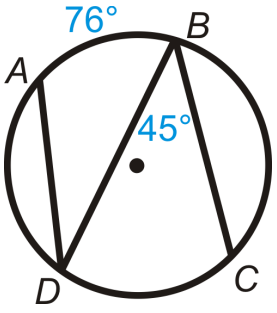
$\angle DAB$ intercepts a semicircle, so $m\angle DAB = 90°$. $\angle DAB$ is a right angle, so $\overset{\frown}{DB}$ is a semicircle.

Example A

Find $m\overset{\frown}{DC}$ and $m\angle ADB$.



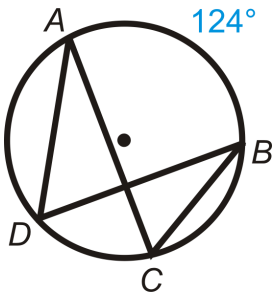
From the Inscribed Angle Theorem:

$$m\overset{\frown}{DC} = 2 \cdot 45° = 90°$$

$$m\angle ADB = \frac{1}{2} \cdot 76° = 38°$$

Example B

Find $m\angle ADB$ and $m\angle ACB$.

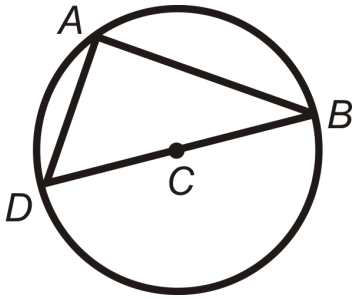


The intercepted arc for both angles is $\widehat{AB}$. Therefore,

$$m\angle ADB = \frac{1}{2} \cdot 124^\circ = 62^\circ$$
$$m\angle ACB = \frac{1}{2} \cdot 124^\circ = 62^\circ$$

Example C

Find $m\angle DAB$ in $\odot C$.



C is the center, so $\overline{DB}$ is a diameter. $\angle DAB$'s endpoints are on the diameter, so the central angle is 180°.

$$m\angle DAB = \frac{1}{2} \cdot 180^\circ = 90^\circ.$$

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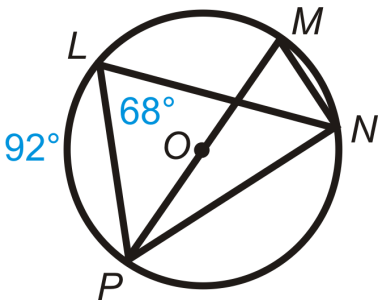
Inscribed Angles in Circles CK-12

Guided Practice

1. Find $m\angle PMN$, $m\widehat{PN}$, $m\angle MNP$, and $m\angle LNP$.



2. Fill in the blank: An inscribed angle is _____ the measure of the intercepted arc.

3. Fill in the blank: A central angle is _____ the measure of the intercepted arc.

Answers:

1. Use what you've learned about inscribed angles.

$$\begin{aligned} m\angle PMN = m\angle PLN &= 68^\circ && \text{by the Congruent Inscribed Angles Theorem.} \\ m\widehat{PN} &= 2 \cdot 68^\circ = 136^\circ && \text{from the Inscribed Angle Theorem.} \\ m\angle MNP &= 90^\circ && \text{by the Semicircle Theorem.} \\ m\angle LNP &= \frac{1}{2} \cdot 92^\circ = 46^\circ && \text{from the Inscribed Angle Theorem.} \end{aligned}$$

2. half

3. equal to

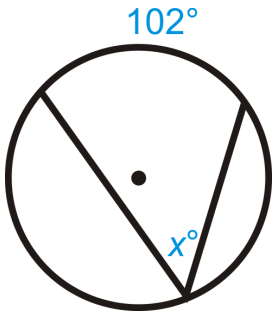
Practice

Fill in the blanks.

1. An angle inscribed in a _____ is 90°.
2. Two inscribed angles that intercept the same arc are _____.
3. The sides of an inscribed angle are _____.
4. Draw inscribed angle $\angle JKL$ in $\odot M$. Then draw central angle $\angle JML$. How do the two angles relate?

Find the value of x and/or y in $\odot A$.

5.



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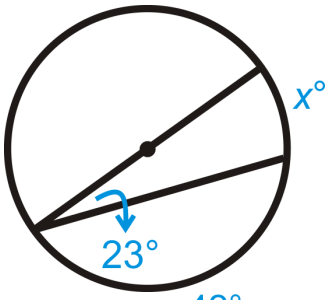

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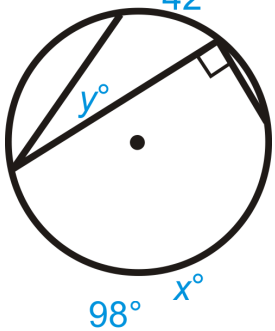

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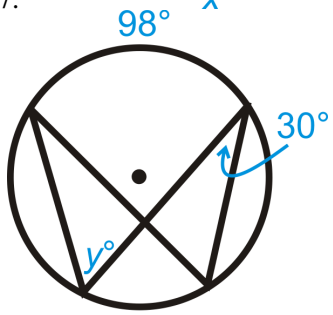

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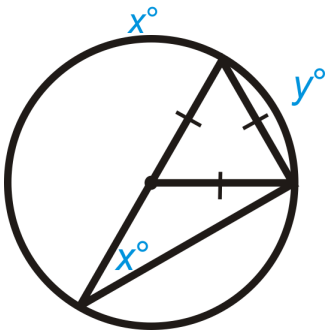
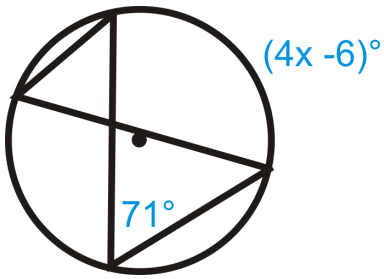


Solve for x.

10.



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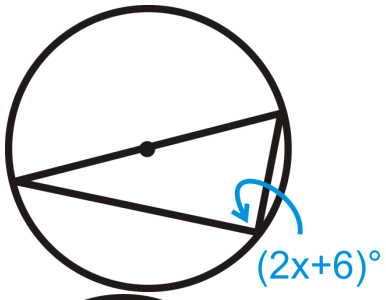

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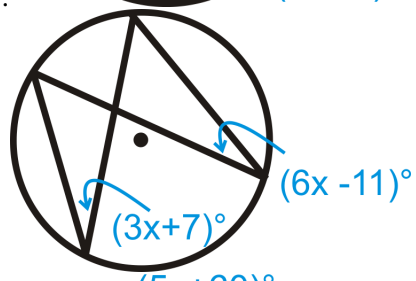

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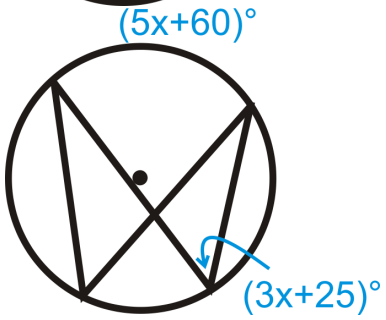
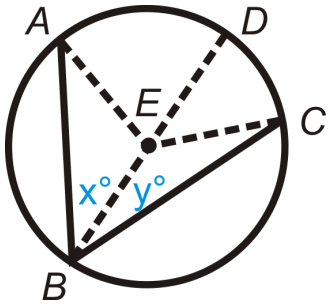


14. Fill in the blanks of the Inscribed Angle Theorem proof.



Given: Inscribed $\angle ABC$ and diameter $\overline{BD}$

Prove: $m\angle ABC = \frac{1}{2}m\widehat{AC}$

TABLE 3.2:

Statement	***Reason***
1. Inscribed $\angle ABC$ and diameter $\overline{BD}$ $m\angle ABE = x^\circ$ and $m\angle CBE = y^\circ$	1.
2. $x^\circ + y^\circ = m\angle ABC$	2.
3.	3. All radii are congruent
4.	4. Definition of an isosceles triangle
5. $m\angle EAB = x^\circ$ and $m\angle ECB = y^\circ$	5.
6. $m\angle AED = 2x^\circ$ and $m\angle CED = 2y^\circ$	6.
7. $m\widehat{AD} = 2x^\circ$ and $m\widehat{DC} = 2y^\circ$	7.
8.	8. Arc Addition Postulate
9. $m\widehat{AC} = 2x^\circ + 2y^\circ$	9.

TABLE 3.2: (continued)

Statement	***Reason***
10.	10. Distributive PoE
11. $m\widehat{AC} = 2m\angle ABC$	11.
12. $m\angle ABC = \frac{1}{2}m\widehat{AC}$	12.

3.5 Inscribed Polygons in Circles

Here you'll learn properties of inscribed quadrilaterals in circles and how to apply them.

What if you were given a circle with a quadrilateral inscribed in it? How could you use information about the arcs formed by the quadrilateral and/or the quadrilateral's angle measures to find the measure of the unknown quadrilateral angles? After completing this Concept, you'll be able to apply the Inscribed Quadrilateral Theorem to solve problems like this one.

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Click image to the left for more content.

CK-12 Foundation: Chapter9InscribedQuadrilateralsinCirclesA

Learn more about cyclic quadrilaterals and parallel lines in circles by watching the video at this link.

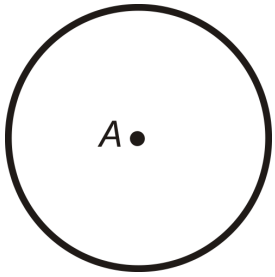
Guidance

An **inscribed polygon** is a polygon where every vertex is on a circle. Note, that not every quadrilateral or polygon can be inscribed in a circle. Inscribed quadrilaterals are also called *cyclic quadrilaterals.* For these types of quadrilaterals, they must have one special property. We will investigate it here.

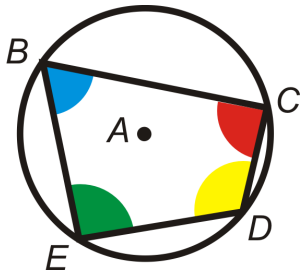
Investigation: Inscribing Quadrilaterals

Tools Needed: pencil, paper, compass, ruler, colored pencils, scissors

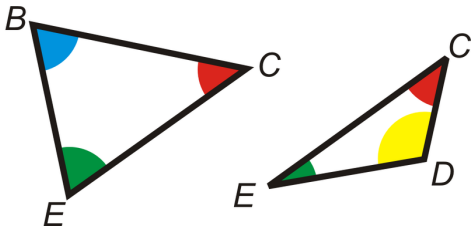
1. Draw a circle. Mark the center point A.



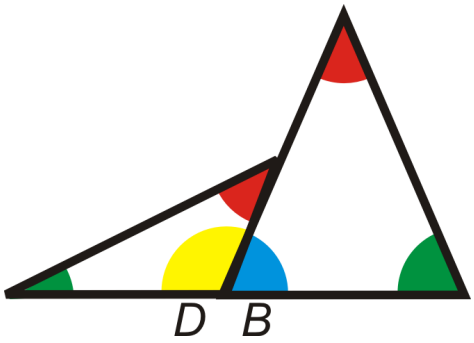
2. Place four points on the circle. Connect them to form a quadrilateral. Color the 4 angles of the quadrilateral 4 different colors.



3. Cut out the quadrilateral. Then cut the quadrilateral into two triangles, by cutting on a diagonal.



4. Line up $\angle B$ and $\angle D$ so that they are adjacent angles. What do you notice? What does this show?



This investigation shows that the opposite angles in an inscribed quadrilateral are supplementary. By cutting the quadrilateral in half, through the diagonal, we were able to show that the other two angles (that we did not cut through) formed a linear pair when matched up.

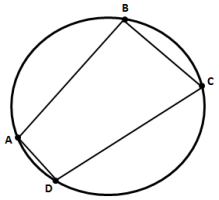
Did you know that a square and a hexagon can also be inscribed in a circle? To watch the videos to create these constructions, click on the links below.

Square Inscribed in a Circle

Hexagon Inscribed in a Circle

Inscribed Quadrilateral Theorem: A quadrilateral is inscribed in a circle if and only if the opposite angles are supplementary.

A quadrilateral that can be inscribed in a circle is also called a **cyclic quadrilateral**. A cyclic quadrilateral (as shown below) has a number of interesting properties:

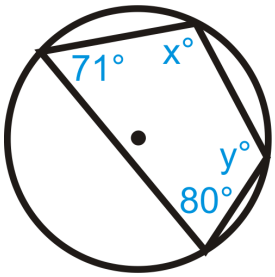


In cyclic quadrilateral *ABCD:*

- $\angle A + \angle C = \angle B + \angle D = 180$ (for proof click here)
- $\angle ABD = \angle ACD$
- $\angle BCA = \angle BDA$
- $\angle BAC = \angle BDC$
- $\angle CAD = \angle CBD$

Example A

Find the value of the missing variable.

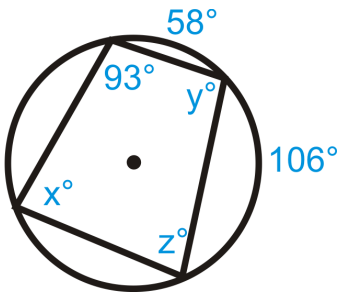


$x + 80^\circ = 180^\circ$ by the Inscribed Quadrilateral Theorem. $x = 100^\circ$.

$y + 71^\circ = 180^\circ$ by the Inscribed Quadrilateral Theorem. $y = 109^\circ$.

Example B

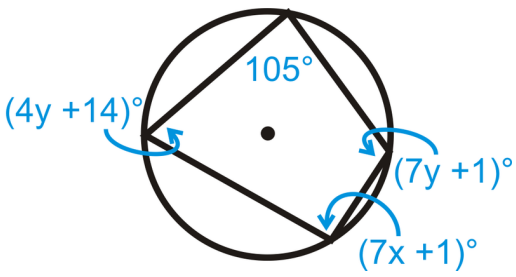
Find the value of the missing variable.



It is easiest to figure out z first. It is supplementary with 93°, so $z = 87^\circ$. Second, we can find x. x is an inscribed angle that intercepts the arc $58^\circ + 106^\circ = 164^\circ$. Therefore, by the Inscribed Angle Theorem, $x = 82^\circ$. y is supplementary with x, so $y = 98^\circ$.Find the value of the missing variables.

Example C

Find x and y in the picture below.



The opposite angles are supplementary. Set up an equation for x and y.

$$(7x+1)^\circ + 105^\circ = 180^\circ$$
$$7x + 106^\circ = 180^\circ$$
$$7x = 84^\circ$$
$$x = 12^\circ$$

$$(4y+14)^\circ + (7y+1)^\circ = 180^\circ$$
$$11y + 15^\circ = 180^\circ$$
$$11y = 165^\circ$$
$$y = 15^\circ$$

Watch this video for help with the Examples above.

MEDIA

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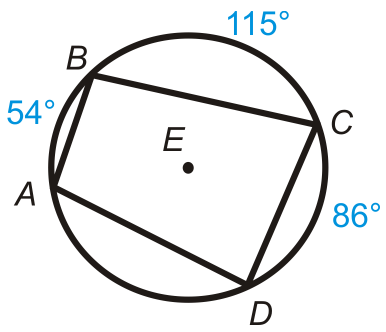
CK-12 Foundation: Chapter9InscribedQuadrilateralsinCirclesB

Vocabulary

A ***circle*** is the set of all points that are the same distance away from a specific point, called the ***center***. A ***radius*** is the distance from the center to the circle. A ***chord*** is a line segment whose endpoints are on a circle. A ***diameter*** is a chord that passes through the center of the circle. The length of a diameter is two times the length of a radius. A ***central angle*** is an angle formed by two radii and whose vertex is at the center of the circle. An ***inscribed angle*** is an angle with its vertex on the circle and whose sides are chords. The ***intercepted arc*** is the arc that is inside the inscribed angle and whose endpoints are on the angle. An ***inscribed polygon*** is a polygon where every vertex is on the circle.

Guided Practice

Quadrilateral $ABCD$ is inscribed in $\odot E$. Find:



1. $m\angle A$
2. $m\angle B$
3. $m\angle C$
4. $m\angle D$

Answers:

First, note that $m\widehat{AD} = 105°$ because the complete circle must add up to 360°.

1. $m\angle A = \frac{1}{2}m\widehat{BD} = \frac{1}{2}(115+86) = 100.5°$
2. $m\angle B = \frac{1}{2}m\widehat{AC} = \frac{1}{2}(86+105) = 95.5°$
3. $m\angle C = 180° - m\angle A = 180° - 100.5° = 79.5°$
4. $m\angle D = 180° - m\angle B = 180° - 95.5° = 84.5°$

Interactive Practice

Multimedia

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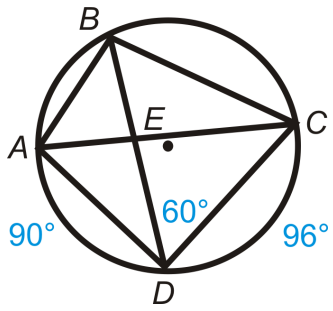
Practice

Fill in the blanks.

1. A(n) _____ polygon has all its vertices on a circle.
2. The _____ angles of an inscribed quadrilateral are _____.

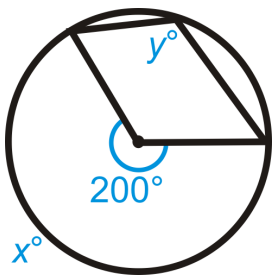
Quadrilateral $ABCD$ is inscribed in $\odot E$. Find:



3. $m\angle DBC$
4. $m\widehat{BC}$
5. $m\widehat{AB}$
6. $m\angle ACD$
7. $m\angle ADC$
8. $m\angle ACB$

Find the value of x and/or y in $\odot A$.

9.



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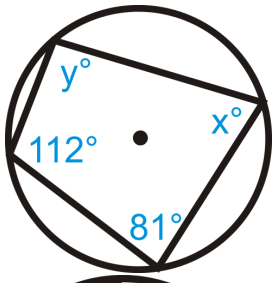

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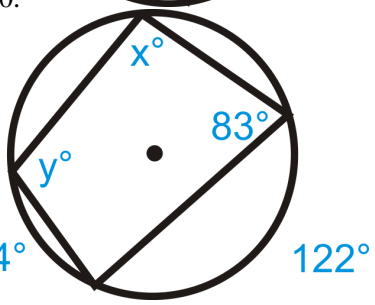
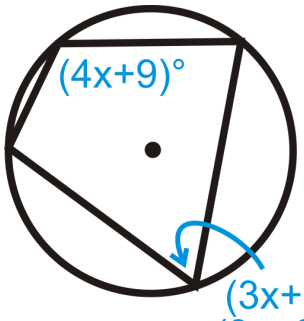
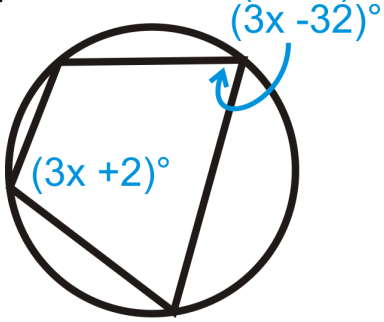


Solve for x.

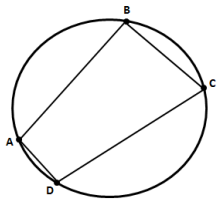
12.



13.



14. Given inscribed quadrilateral $ABCD$ (see below), prove that $\angle A$ is supplementary to $\angle C$.
15. Given inscribed quadrilateral $ABCD$ (see below), prove that $\angle BCA = \angle BDA$.



3.6 Segments from Secants

Here you'll learn the relationship between the segments that are created when two secants intersect outside a circle. You'll then use this relationship to solve for unknown circle values.

What if you were given a circle with two secants that intersect outside the circle? How could you use the length of some of the segments formed by their intersection to determine the lengths of the unknown segments? After completing this Concept, you'll be able to use the Two Secants Segments Theorem to solve problems like this one.

Watch This

MEDIA

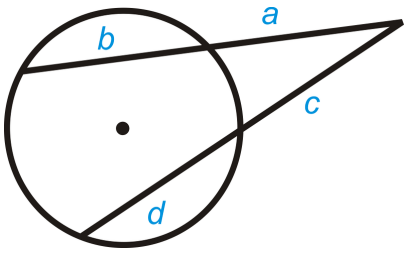
Click image to the left for more content.

Segments from Secants CK-12

Guidance

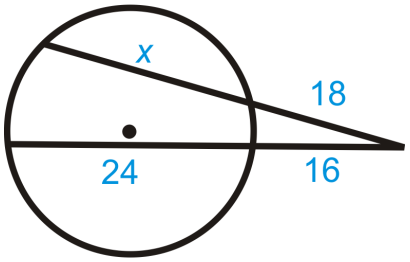
When two secants intersect outside a circle, the circle divides the secants into segments that are proportional with each other.

Two Secants Segments Theorem: If two secants are drawn from a common point outside a circle and the segments are labeled as below, then $a(a+b) = c(c+d)$.



Example A

Find the value of x.

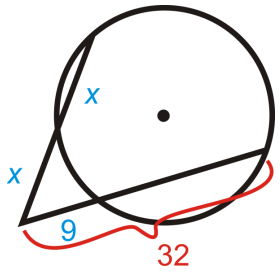


Use the Two Secants Segments Theorem to set up an equation.

$$\begin{aligned} 18 \cdot (18+x) &= 16 \cdot (16+24) \\ 324+18x &= 256+384 \\ 18x &= 316 \\ x &= 17\frac{5}{9} \end{aligned}$$

Example B

Find the value of x.



Use the Two Secants Segments Theorem to set up an equation.

$$\begin{aligned} x \cdot (x+x) &= 9 \cdot 32 \\ 2x^2 &= 288 \\ x^2 &= 144 \\ x &= 12,\ x \neq -12 \text{ (length is not negative)} \end{aligned}$$

Example C

True or False: Two secants will always intersect outside of a circle.

This is false. If the two secants are parallel, they will never intersect. It's also possible for two secants to intersect inside a circle.

MEDIA

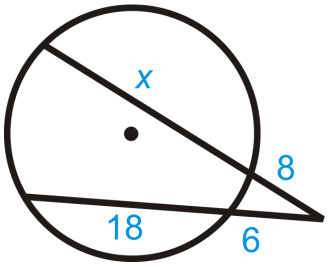
Click image to the left for more content.

Segments from Secants CK-12

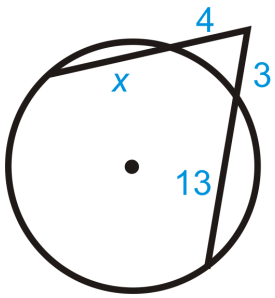
Guided Practice

Find x in each diagram below. Simplify any radicals.

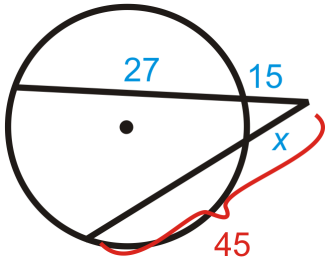
1.



2.



3.



Answers:

Use the Two Secants Segments Theorem.

1.

$$\begin{aligned} 8(8+x) &= 6(6+18) \\ 64+8x &= 144 \\ 8x &= 80 \\ x &= 10 \end{aligned}$$

2.

$$\begin{aligned} 4(4+x) &= 3(3+13) \\ 16+4x &= 48 \\ 4x &= 32 \\ x &= 8 \end{aligned}$$

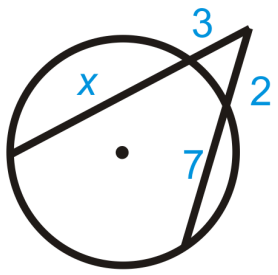
3.

$$15(15+27) = x \cdot 45$$
$$630 = 45x$$
$$x = 14$$

Practice

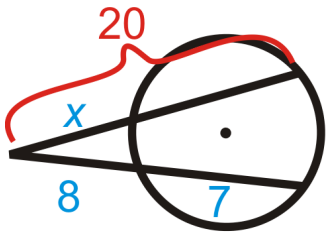
Fill in the blanks for each problem below. Then, solve for the missing segment.

1.



$3(___ + ___) = 2(2+7)$

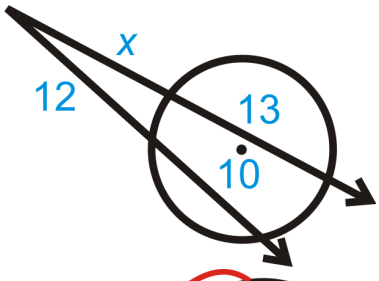
2.



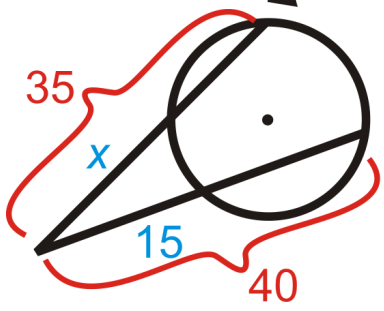
$x \cdot ___ = 8(___ + ___)$

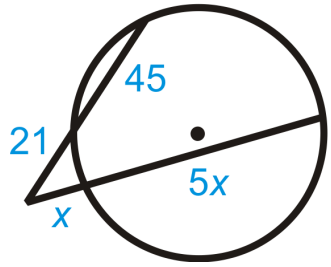
Find x in each diagram below. Simplify any radicals.

3.



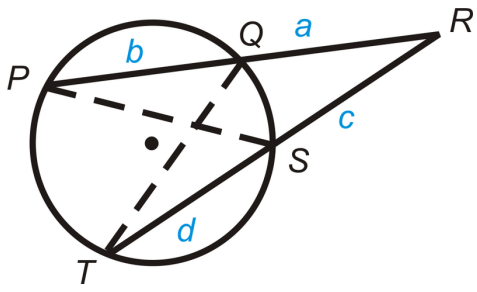
4.





5.

6. Fill in the blanks of the proof of the Two Secants Segments Theorem.



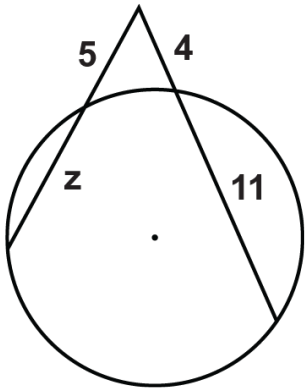
Given: Secants $\overline{PR}$ and $\overline{RT}$

Prove: $a(a+b)=c(c+d)$

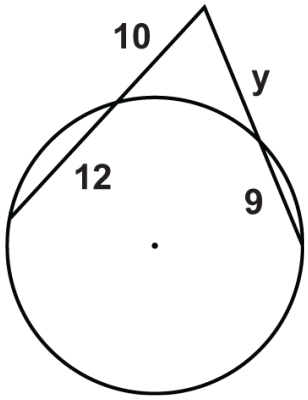
TABLE 3.3:

Statement	***Reason***
1. Secants $\overline{PR}$ and $\overline{RT}$ with segments a, b, c, and d.	1. Given
2. $\angle R \cong \angle R$	2. Reflexive PoC
3. $\angle QPS \cong \angle STQ$	3. Congruent Inscribed Angles Theorem
4. $\triangle RPS \sim \triangle RTQ$	4. AA Similarity Postulate
5. $\frac{a}{c+d}=\frac{c}{a+b}$	5. Corresponding parts of similar triangles are proportional
6. $a(a+b)=c(c+d)$	6. Cross multiplication

Solve for the unknown variable.



7.



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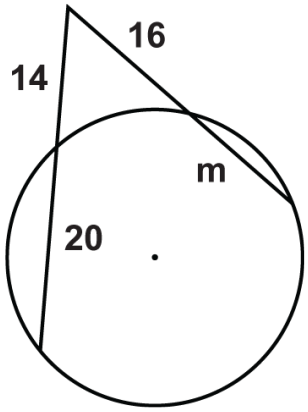
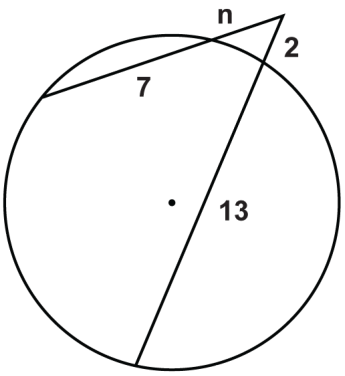
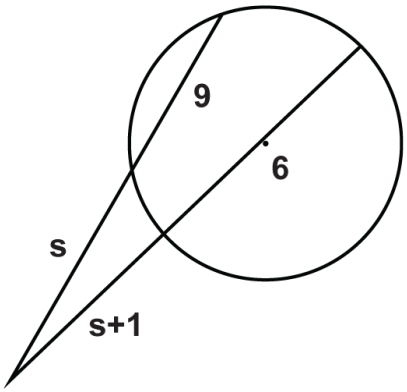


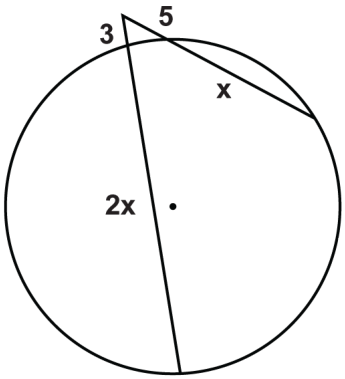
9.



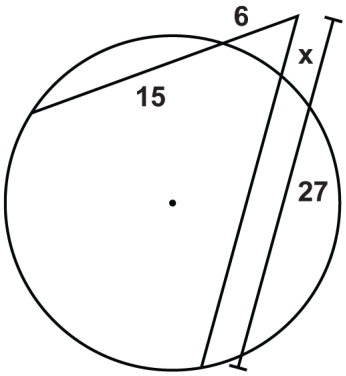
10.



11.



12.



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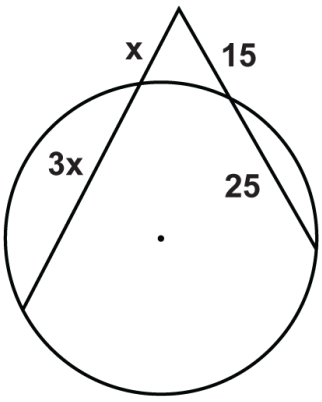
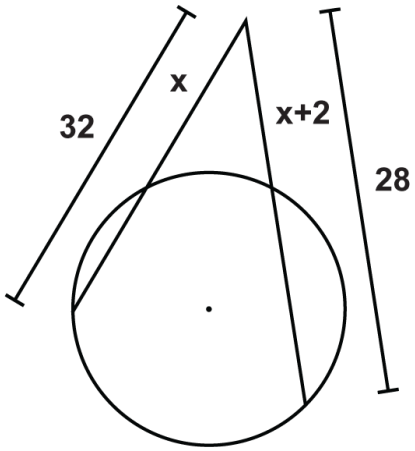


14.



15.

3.7 Segments from Secants and Tangents

Here you'll learn the relationship that exists between the segments that are created when a secant and tangent intersect on a circle. You'll then use this relationship to solve for unknown circle values.

What if you were given a circle with a tangent and a secant that intersect outside the circle? How could you use the length of some of the segments formed by their intersection to determine the lengths of the unknown segments? After completing this Concept, you'll be able to use the Tangent Secant Segment Theorem to solve problems like this one.

Watch This

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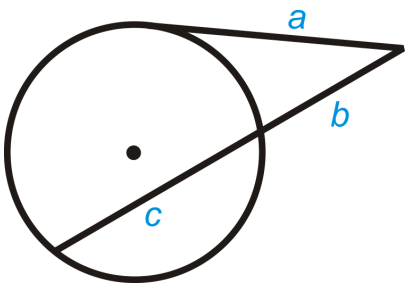
Click image to the left for more content.

Segments from Secants and Tangents CK-12

Guidance

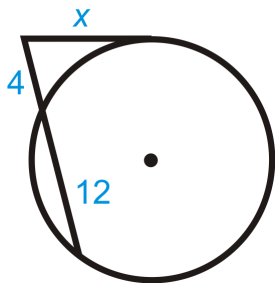
If a tangent and secant meet at a common point outside a circle, the segments created have a similar relationship to that of two secant rays.

Tangent Secant Segment Theorem: If a tangent and a secant are drawn from a common point outside the circle (and the segments are labeled like the picture below), then $a^2 = b(b+c)$.



Example A

Find the length of the missing segment.

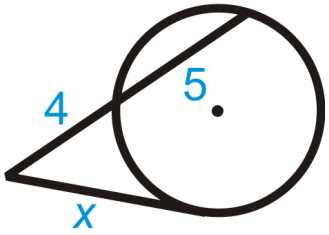


Use the Tangent Secant Segment Theorem.

$$x^2 = 4(4+12)$$
$$x^2 = 4 \cdot 16 = 64$$
$$x = 8$$

Example B

Fill in the blank and then solve for the missing segment.

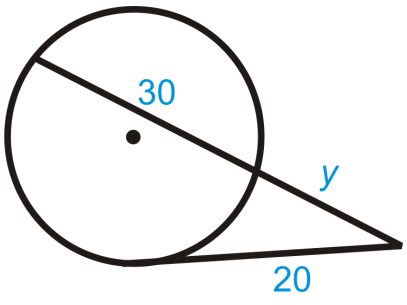


___ = ___(4 + 5)

$$x^2 = 4(4+5)$$
$$x^2 = 36$$
$$x = 6$$

Example C

Find the value of the missing segment.



Use the Tangent Secant Segment Theorem.

$$\begin{aligned} 20^2 &= y(y+30) \\ 400 &= y^2+30y \\ 0 &= y^2+30y-400 \\ 0 &= (y+40)(y-10) \\ y &= \cancel{-40}, 10 \end{aligned}$$

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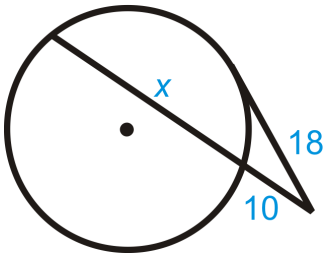
Segments from Secants and Tangents CK-12

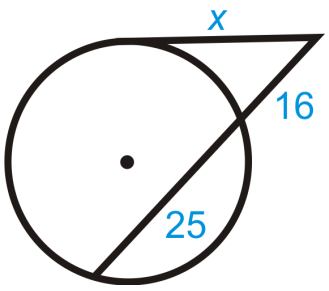
Guided Practice

Find x in each diagram below. Simplify any radicals.

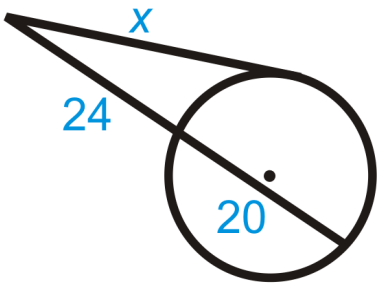
1.



2.



3.



Answers:

Use the Tangent Secant Segment Theorem.

1.

$$18^2 = 10(10+x)$$
$$324 = 100+10x$$
$$224 = 10x$$
$$x = 22.4$$

2.

$$x^2 = 16(16+25)$$
$$x^2 = 656$$
$$x = 4\sqrt{41}$$

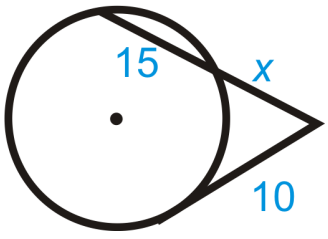
3.

$$x^2 = 24(24+20)$$
$$x^2 = 1056$$
$$x = 4\sqrt{66}$$

Practice

Fill in the blanks for each problem below and then solve for the missing segment.

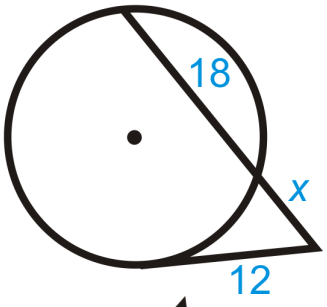
1.



$10^2 = x(___ + ___)$

Find x in each diagram below. Simplify any radicals.

2.



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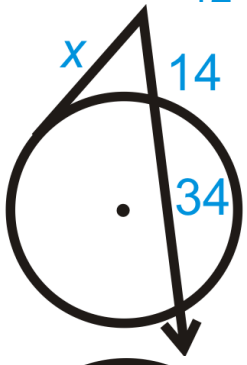

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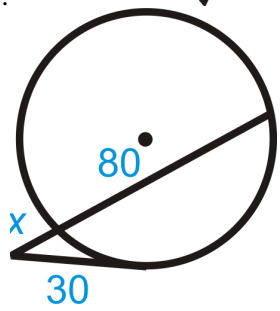
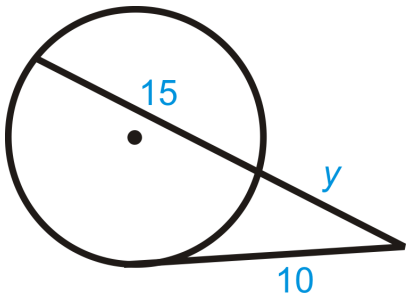
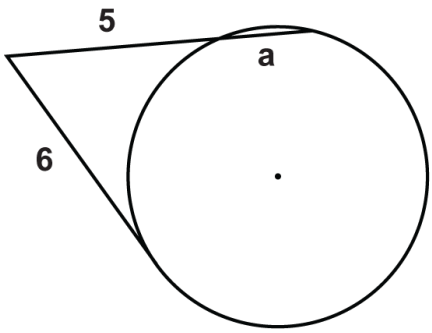


5. Describe and correct the error in finding y.



$$10 \cdot 10 = y \cdot 15y$$
$$100 = 15y^2$$
$$\frac{20}{3} = y^2$$
$$\frac{2\sqrt{15}}{3} = y \quad \longleftarrow \quad y \text{ is not correct}$$

Solve for the unknown variable.



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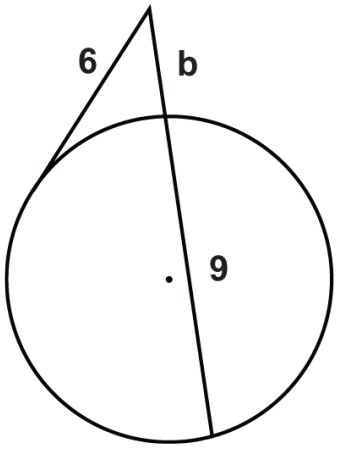

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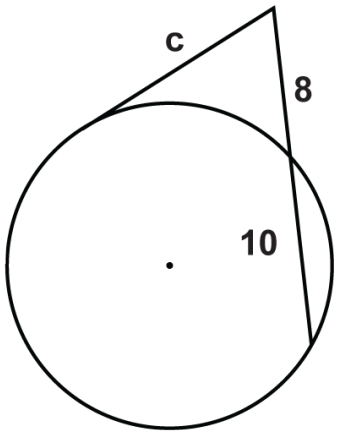

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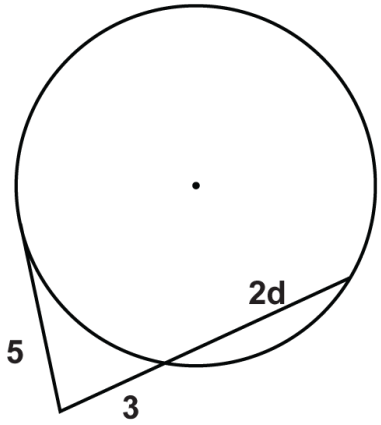

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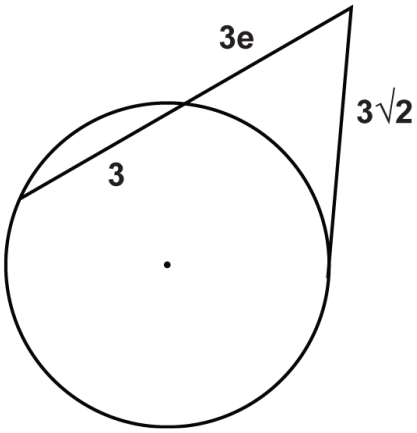

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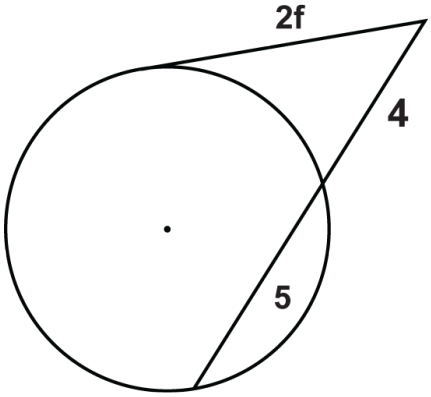

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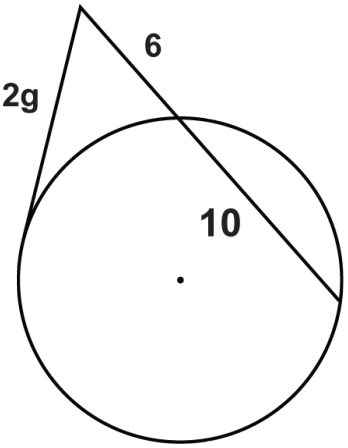

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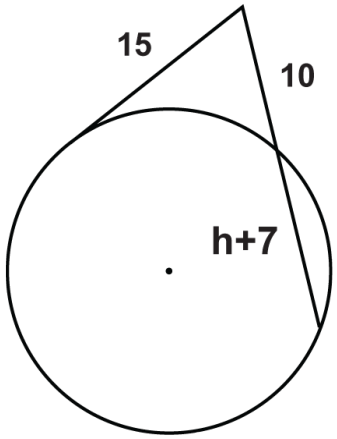

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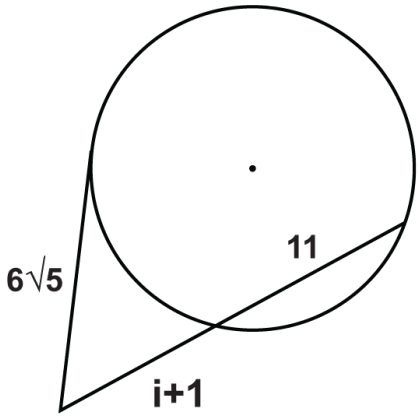

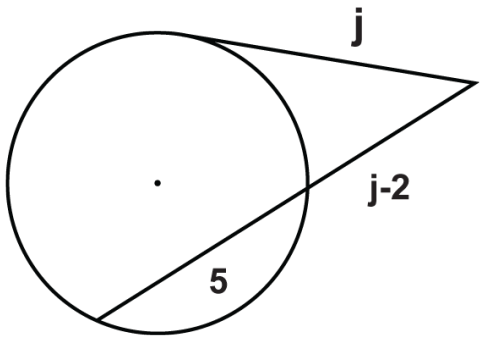

15.

3.8 Angles On and Inside a Circle

Here you'll learn how to solve problems containing angles that are on or inside a circle.

What if you were given a circle with either a chord and a tangent or two chords that meet at a common point? How could you use the measure of the arc(s) formed by those circle parts to find the measure of the angles they make on or inside the circle? After completing this Concept, you'll be able to apply the Chord/Tangent Angle Theorem and the Intersecting Chords Angle Theorem to solve problems like this one.

Watch This

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Click image to the left for more content.

CK-12 Foundation: Chapter9AnglesOnandInsideaCircleA

Learn more about chords and tangents by watching the second part of the video at this link.

Follow this link to watch a video about secants.

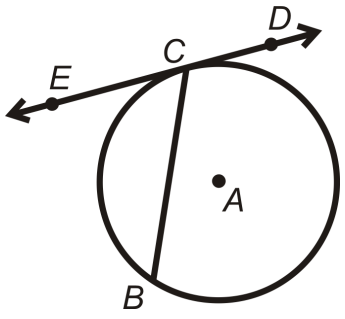
Guidance

When an angle is on a circle, the vertex is on the circumference of the circle. One type of angle *on* a circle is one formed by a tangent and a chord.

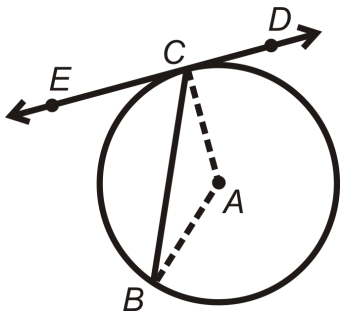
Investigation: The Measure of an Angle formed by a Tangent and a Chord

Tools Needed: pencil, paper, ruler, compass, protractor

1. Draw $\odot A$ with chord $\overline{BC}$ and tangent line $\overleftrightarrow{ED}$ with point of tangency C.



2. Draw in central angle $\angle CAB$. Then, using your protractor, find $m\angle CAB$ and $m\angle BCE$.



3. Find $m\widehat{BC}$ (the minor arc). How does the measure of this arc relate to $m\angle BCE$?

This investigation proves the Chord/Tangent Angle Theorem.

Chord/Tangent Angle Theorem: The measure of an angle formed by a chord and a tangent that intersect on the circle is half the measure of the intercepted arc.

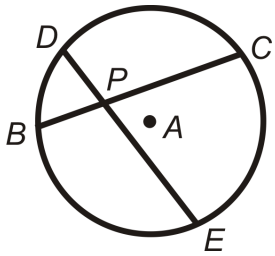
From the Chord/Tangent Angle Theorem, we now know that there are two types of angles that are half the measure of the intercepted arc; an inscribed angle and an angle formed by a chord and a tangent. Therefore, ***any angle with its vertex on a circle will be half the measure of the intercepted arc***.

An angle is considered *inside* a circle when the vertex is somewhere inside the circle, but not on the center. All angles inside a circle are formed by two intersecting chords.

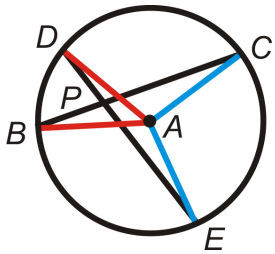
Investigation: Find the Measure of an Angle

Tools Needed: pencil, paper, compass, ruler, protractor, colored pencils (optional)

1. Draw $\odot A$ with chord $\overline{BC}$ and $\overline{DE}$. Label the point of intersection P.



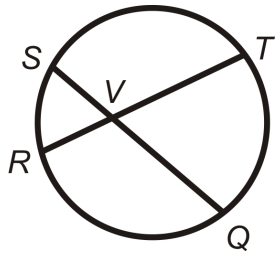
2. Draw central angles $\angle DAB$ and $\angle CAE$. Use colored pencils, if desired.



3. Using your protractor, find $m\angle DPB, m\angle DAB$, and $m\angle CAE$. What is $m\widehat{DB}$ and $m\widehat{CE}$?
4. Find $\frac{m\widehat{DB}+m\widehat{CE}}{2}$.
5. What do you notice?

Intersecting Chords Angle Theorem: The measure of the angle formed by two chords that intersect *inside* a circle is the average of the measure of the intercepted arcs.

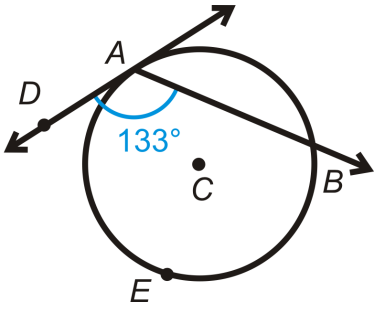
In the picture below:



$$m\angle SVR = \frac{1}{2}\left(m\widehat{SR} + m\widehat{TQ}\right) = \frac{m\widehat{SR} + m\widehat{TQ}}{2} = m\angle TVQ$$
$$m\angle SVT = \frac{1}{2}\left(m\widehat{ST} + m\widehat{RQ}\right) = \frac{m\widehat{ST} + m\widehat{RQ}}{2} = m\angle RVQ$$

Example A

Find $m\widehat{AEB}$

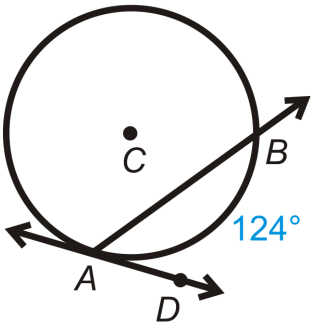


Use the Chord/Tangent Angle Theorem.

$m\widehat{AEB} = 2 \cdot m\angle DAB = 2 \cdot 133^\circ = 266^\circ$

Example B

Find $m\angle BAD$.

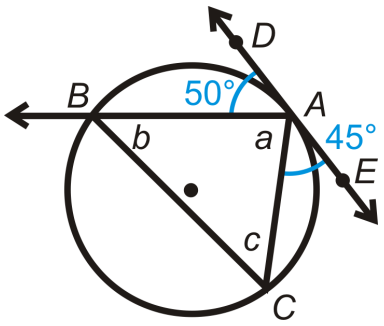


Use the Chord/Tangent Angle Theorem.

$m\angle BAD = \frac{1}{2}m\widehat{AB} = \frac{1}{2} \cdot 124^\circ = 62^\circ$

Example C

Find a, b, and c.



To find a, it is in line with 50° and 45°. The three angles add up to 180°. $50^\circ + 45^\circ + m\angle a = 180^\circ, m\angle a = 85^\circ$.

b is an inscribed angle, so its measure is half of $m\widehat{AC}$. From the Chord/Tangent Angle Theorem, $m\widehat{AC} = 2 \cdot m\angle EAC = 2 \cdot 45^\circ = 90^\circ$.

$m\angle b = \frac{1}{2} \cdot m\widehat{AC} = \frac{1}{2} \cdot 90^\circ = 45^\circ$.

To find c, you can either use the Triangle Sum Theorem or the Chord/Tangent Angle Theorem. We will use the Triangle Sum Theorem. $85^\circ + 45^\circ + m\angle c = 180^\circ, m\angle c = 50^\circ$.

Watch this video for help with the Examples above.

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CK-12 Foundation: Chapter9AnglesOnandInsideaCircleB

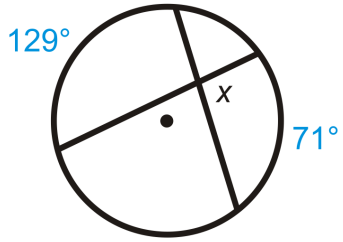
Vocabulary

A ***circle*** is the set of all points that are the same distance away from a specific point, called the ***center***. A ***radius*** is the distance from the center to the circle. A ***chord*** is a line segment whose endpoints are on a circle. A ***diameter*** is a chord that passes through the center of the circle. The length of a diameter is two times the length of a radius. A ***central angle*** is the angle formed by two radii and whose vertex is at the center of the circle. An ***inscribed angle*** is an angle with its vertex on the circle and whose sides are chords. The ***intercepted arc*** is the arc that is inside the inscribed angle and whose endpoints are on the angle. A ***tangent*** is a line that intersects a circle in exactly one point. The ***point of tangency*** is the point where the tangent line touches the circle.

Guided Practice

Find x.

1.



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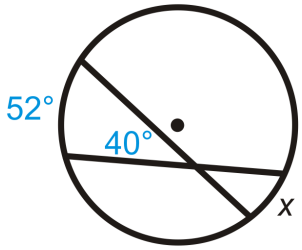

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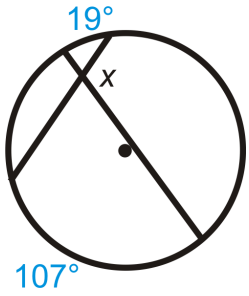

Answers:

Use the Intersecting Chords Angle Theorem and write an equation.

1. The intercepted arcs for x are 129° and 71°.

$$x = \frac{129^\circ + 71^\circ}{2} = \frac{200^\circ}{2} = 100^\circ$$

2. Here, x is one of the intercepted arcs for 40°.

$$\begin{aligned} 40^\circ &= \frac{52^\circ + x}{2} \\ 80^\circ &= 52^\circ + x \\ 38^\circ &= x \end{aligned}$$

3. x is supplementary to the angle that the average of the given intercepted arcs. We will call this supplementary angle y.

$y = \frac{19^\circ + 107^\circ}{2} = \frac{126^\circ}{2} = 63^\circ$ This means that $x = 117^\circ; 180^\circ - 63^\circ$

Interactive Practice

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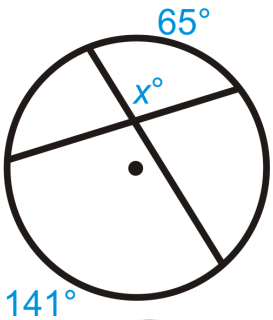
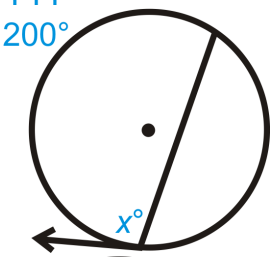
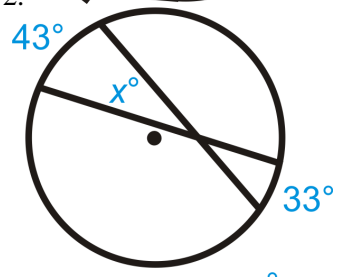
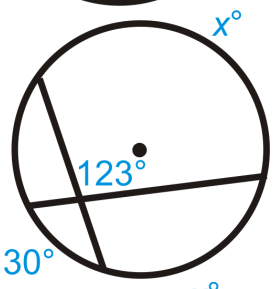
Practice

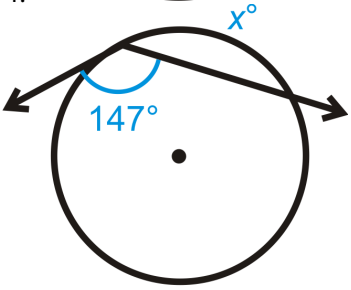
Find the value of the missing variable(s).

1. 

2. 

3. 

4. 

5. 

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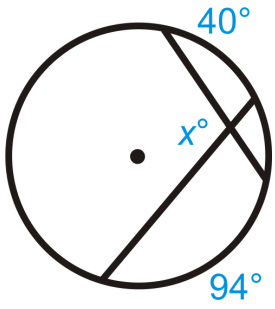
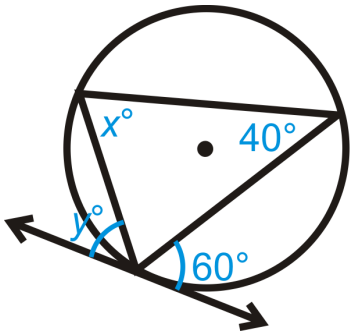
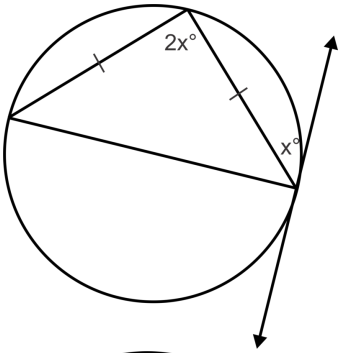


7. $y \neq 60°$



Solve for x.

8.



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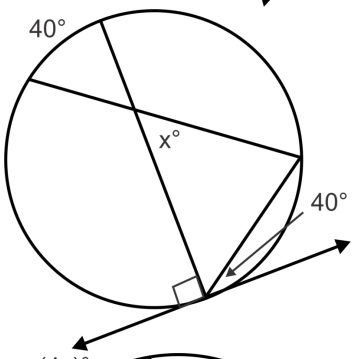

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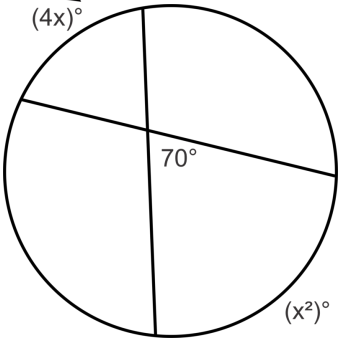
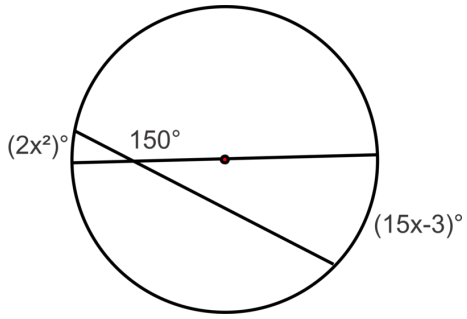
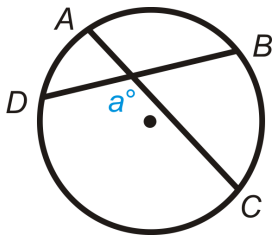


11. 

12. Prove the Intersecting Chords Angle Theorem.



Given: Intersecting chords $\overline{AC}$ and $\overline{BD}$.

Prove: $m\angle a = \frac{1}{2}\left(m\widehat{DC} + m\widehat{AB}\right)$

Fill in the blanks.

13. If the vertex of an angle is _____ a circle, then its measure is the average of the _____-_____ arcs.
14. If the vertex of an angle is _____ a circle, then its measure is _____ the intercepted arc.
15. Can two tangent lines intersect inside a circle? Why or why not?

3.9 Circles Similarity & Arc Length

Here you'll learn how to find the length of an arc.

What if you wanted to find the "length" of the crust for an individual slice of pizza? A typical large pizza has a diameter of 14 inches and is cut into 8 or 10 pieces. If the "length" of the entire crust is the circumference of the pizza, find the "length" of the crust for one piece of pizza when the entire pizza is cut into a) 8 pieces or b) 10 pieces. After completing this Concept, you'll be able to answer these questions.

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CK-12 Foundation: Chapter10ArcLengthA

Learn more about arc length by watching the video at this link.

Guidance

The **circumference** of a circle is the distance around a circle. The number **pi** (π) is the ratio of the circumference of a circle to its diameter. The circumference of a circle is π times the diameter: $C = \pi d$. Since the diameter is also equal to two times the radius, circumference can also be found with the following: $C = 2\pi r$.

Are all circles similar? Think back to previous work you have done with similarity transformations. If a similarity transformation will map one circle on to another, the two circles are similar. Consider the two circles in Figure 3.1. What similarity transformation will map circle A to circle B?

A translation will map circle A so that it shares the same center with circle B. A dilation with scale factor $\frac{k}{h}$ will map circle A to circle B. Recall that a translation followed by a dilation is a similarity transformation. Since a similarity transformation maps circle A to circle B, the two circles are similar.

Coplanar circles that have the same center are called **concentric circles**.

The length of an arc is traditionally denoted with an s.

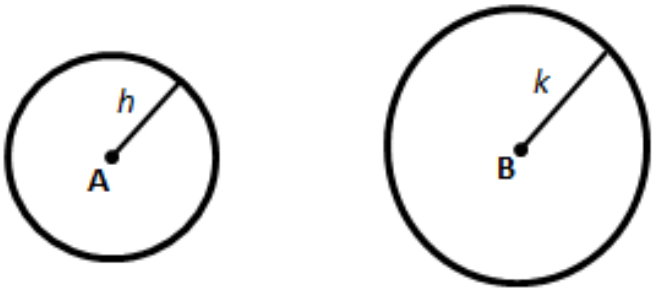


FIGURE 3.1

One way to measure arcs is in degrees. This is called the "arc measure" or "degree measure." Arcs can also be measured in length, as a portion of the circumference. **Arc length** is the length of an arc or a portion of a circle's circumference. The arc length is directly related to the degree arc measure.

The measure of an arc is in degrees. The arc length is a fraction of the circumference of a circle.

Consider the arcs shown below. Since the circles are concentric, a dilation will map circle A to circle B. The same dilation will map the green slice of the circle A to the blue slice of the circle B. Since corresponding lengths of similar figures are proportional,

$\frac{r_1}{r_2} = \frac{s_1}{s_2}$
$r_1 s_2 = r_2 s_1$
$s_1 = r_1 \frac{s_2}{r_2}$

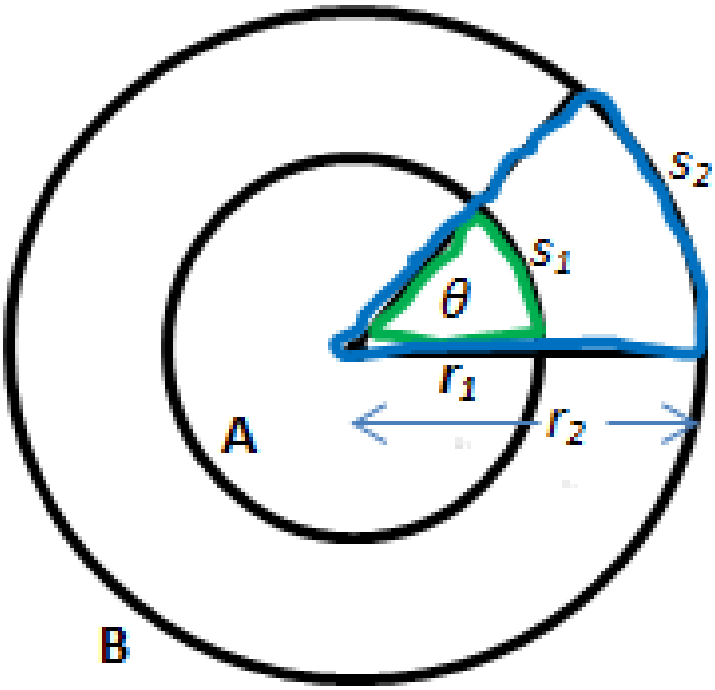


FIGURE 3.2

This means that the arc length s_1 is equal to the radius r_1 times some number. This means that the length of the arc

intercepted by an angle is proportional to the radius.

You are familiar with measuring angles in degrees. You can also measure angles in **radians**. In the previous figure, you saw that central angle θ of a circle intercepts an arc whose length s is proportional to the radius r. The ratio $\frac{s}{r}$, which is the constant of proportionality, is defined as the measure of θ in radians.

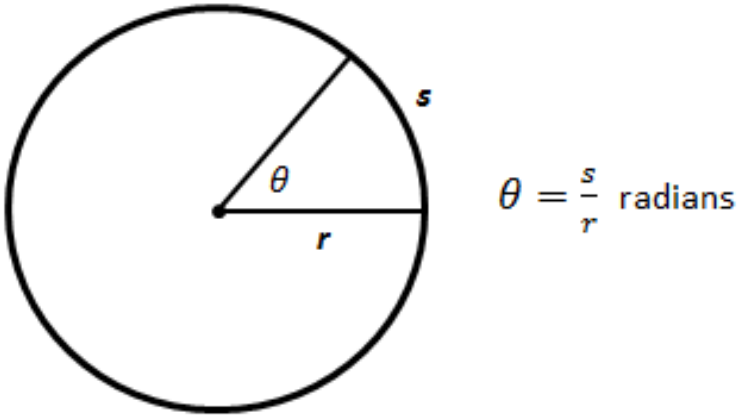


FIGURE 3.3

When the arc length equals an entire circumference, we can use

$s = \theta r$
$2\pi r = \theta r$
$2\pi = 360^\circ$

Due to this relationship, to change from degrees to radians, multiply by $\frac{\pi}{180^\circ}$. To change from radians to degrees, multiply by $\frac{180^\circ}{\pi}$.

Example A

Copy and complete the table.

TABLE 3.4:

Circle Part	Degree Measure	Radian Measure
Full Circle		
Half Circle		
Quarter Circle		
One-Eighth Circle		
$\frac{1}{360}$ of a Circle		

To fill in the table, multiply each degree measure by $\frac{\pi}{180}$.

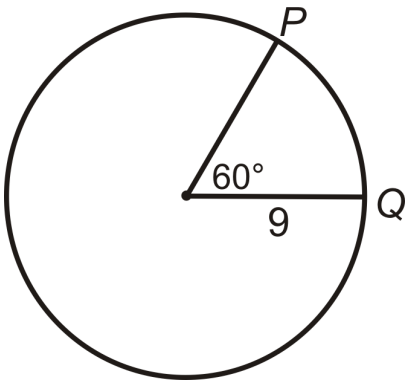
TABLE 3.5:

Circle Part	Degree Measure	Radian Measure
Full Circle	360°	2π radians
Half Circle	180°	π radians
Quarter Circle	90°	$\frac{\pi}{2}$ radians
One-Eighth Circle	45°	$\frac{\pi}{4}$ radians
$\frac{1}{360}$ of a Circle	1°	$\frac{\pi}{180}$ radians

Arc Length Formula: If d is the diameter or r is the radius, the length of $\widehat{AB} = \frac{m\widehat{AB}}{360^\circ} \cdot \pi d$ or $\frac{m\widehat{AB}}{360^\circ} \cdot 2\pi r$.

Example B

Find the length of $\widehat{PQ}$. Leave your answer in terms of π.



In the picture, the central angle that corresponds with $\widehat{PQ}$ is 60°. This means that $m\widehat{PQ} = 60^\circ$ as well. So, think of the arc length as a portion of the circumference. There are 360° in a circle, so 60° would be $\frac{1}{6}$ of that $\left(\frac{60^\circ}{360^\circ} = \frac{1}{6}\right)$. Therefore, the length of $\widehat{PQ}$ is $\frac{1}{6}$ of the circumference.

$$length\ of\ \widehat{PQ} = \frac{1}{6} \cdot 2\pi(9) = 3\pi$$

Example C

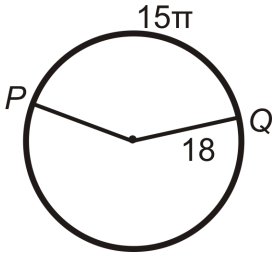
The arc length of $\widehat{AB} = 6\pi$ and is $\frac{1}{4}$ the circumference. Find the radius of the circle.

If 6π is $\frac{1}{4}$ the circumference, then the total circumference is $4(6\pi) = 24\pi$. To find the radius, plug this into the circumference formula and solve for r.

$$24\pi = 2\pi r$$
$$12 = r$$

Example D

Find the measure of the central angle or $\widehat{PQ}$.



Let's plug in what we know to the Arc Length Formula.

$$15\pi = \frac{m\widehat{PQ}}{360^\circ} \cdot 2\pi(18)$$
$$15 = \frac{m\widehat{PQ}}{10^\circ}$$
$$150^\circ = m\widehat{PQ}$$

Watch this video for help with the Examples above.

Multimedia

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Click image to the left for more content.

CK-12 Foundation: Chapter10ArcLengthB

For more information on arc length and radian measure, click here.

For a video that highlights these concepts, click here. (click the (x) in the top of the video window to access the video).

Concept Problem Revisited

In the picture below, the top piece of pizza is if it is cut into 8 pieces. Therefore, for $\frac{1}{8}$ of the pizza, one piece would have $\frac{44}{8} \approx 5.5$ *inches* of crust. The bottom piece of pizza is if the pizza is cut into 10 pieces. For $\frac{1}{10}$ of the crust, one piece would have $\frac{44}{10} \approx 4.4$ *inches* of crust.

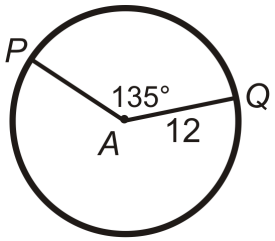
Vocabulary

Circumference is the distance around a circle. ***Pi*** (π) is the ratio of the circumference of a circle to its diameter. **Concentric circles** are coplanar circles that have the same center. ***Arc length*** is the length of an arc or a portion of a circle's circumference. A **radian**, like degrees, is a unit for measuring angles.

Guided Practice

Find the arc length of $\widehat{PQ}$ in $\odot A$. Leave your answers in terms of π.

1.



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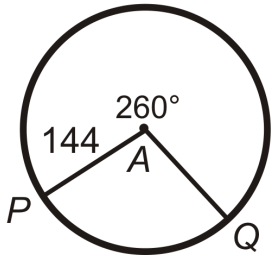

3. An extra large pizza has a diameter of 20 inches and is cut into 12 pieces. Find the length of the crust for one piece of pizza.

Answers:

1. Use the Arc Length formula.

$$\widehat{PQ} = \frac{135}{360} \cdot 2\pi(12)$$
$$\widehat{PQ} = \frac{3}{8} \cdot 24\pi$$
$$\widehat{PQ} = 9\pi$$

2. Use the Arc Length formula.

$$\widehat{PQ} = \frac{360 - 260}{360} \cdot 2\pi(144)$$
$$\widehat{PQ} = \frac{5}{18} \cdot 288\pi$$
$$\widehat{PQ} = 80\pi$$

3. The entire length of the crust, or the circumference of the pizza, is $20\pi \approx 62.83$ *in*. In $\frac{1}{12}$ of the pizza, one piece would have $\frac{62.83}{12} \approx 5.24$ inches of crust.

Interactive Practice

Multimedia

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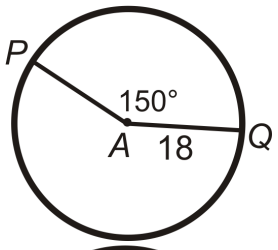
Practice

Find the arc length of $\widehat{PQ}$ in $\odot A$. Leave your answers in terms of π.

1.



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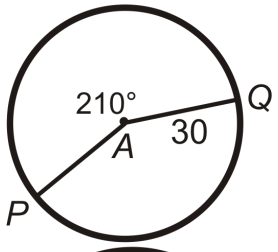

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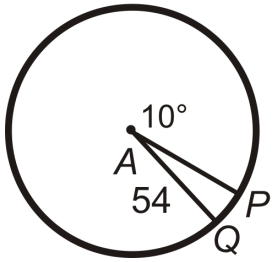

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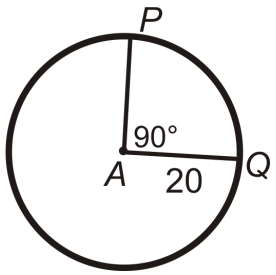
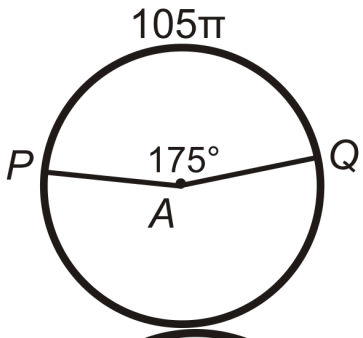


Find PA (the radius) in $\odot A$. Leave your answer in terms of π.

5.



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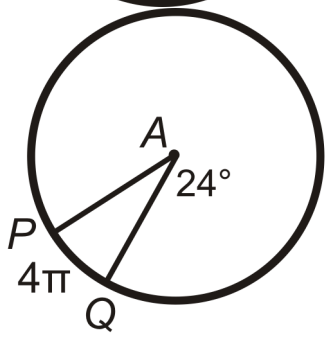

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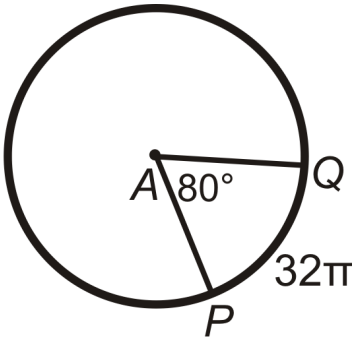
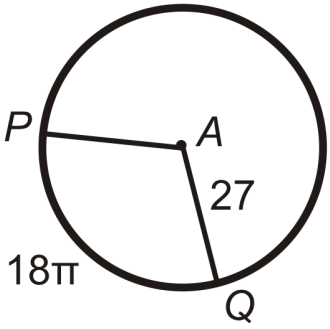


Find the central angle or $m\widehat{PQ}$ in $\odot A$. Round any decimal answers to the nearest tenth.

8.



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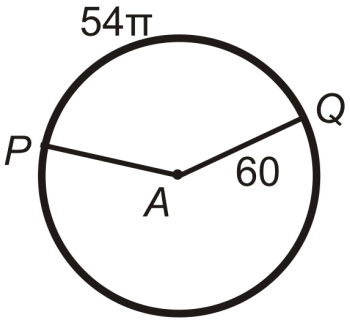

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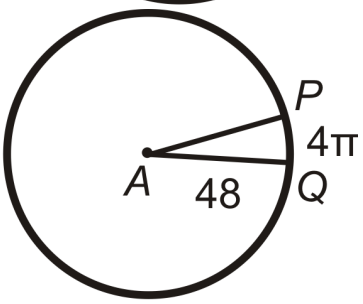
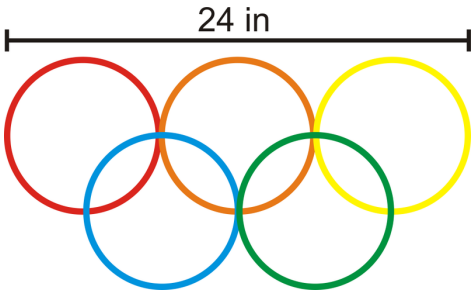


11. The Olympics symbol is five congruent circles arranged as shown below. Assume the top three circles are tangent to each other. Brad is tracing the entire symbol for a poster. How far will his pen point travel?



Mario's Pizza Palace offers a stuffed crust pizza in three sizes (diameter length) for the indicated prices:

The Little Cheese, 8 in, $7.00

The Big Cheese, 10 in, $9.00

The Cheese Monster, 12 in, $12.00

12. What is the crust (in) to price ($) ratio for The Little Cheese?
13. What is the crust (in) to price ($) ratio for The Little Cheese?
14. What is the crust (in) to price ($) ratio for The Little Cheese?
15. Michael thinks the cheesy crust is the best part of the pizza and wants to get the most crust for his money. Which pizza should he buy?
16. What is the difference between the measure of an arc and arc length? Explain.
17. In a circle with radius 1.5 units, what is the length of an arc intercepted by a central angle that measures 2.3 radians?
18. Are all circles similar? Are all circles congruent? Explain.

3.10 Area of Sectors and Segments

Here you'll learn how to find the area of a sector or a segment of a circle.

What if you wanted to find the area of a pizza, this time taking into consideration the area of the crust? In another Concept, we found the length of the crust for a 14 in pizza. However, crust typically takes up some area on a pizza. Leave your answers in terms of π and reduced improper fractions.

a) Find the area of the crust of a deep-dish 16 in pizza. A typical deep-dish pizza has 1 in of crust around the toppings.

b) A thin crust pizza has $\frac{1}{2}$ - in of crust around the edge of the pizza. Find the area of a thin crust 16 in pizza.

c) Which piece of pizza has more crust? A twelfth of the deep dish pizza or a fourth of the thin crust pizza?

After completing this Concept, you will be able to answer these questions.

Watch This

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Click image to the left for more content.

CK-12 Foundation: Chapter10AreaofSectorsandSegmentsA

Learn more about the area of a sector by watching the video at this link.

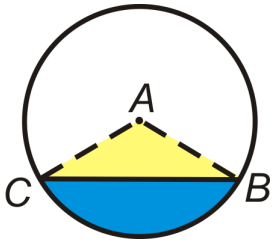
Watch this video to learn even more about the area of a sector.

Guidance

A **sector of a circle** is the area bounded by two radii and the arc between the endpoints of the radii.

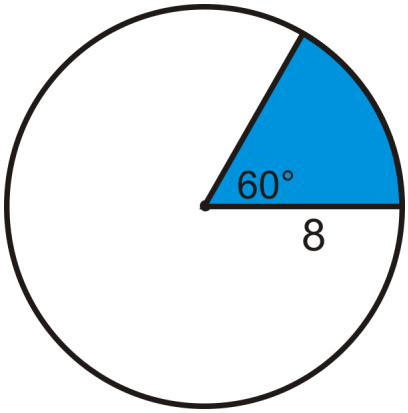
The area of a sector is a fractional part of the area of the circle, just like arc length is a fractional portion of the circumference. The **Area of a sector** is $A = \frac{m\widehat{AB}}{360^\circ} \cdot \pi r^2$ where r is the radius and $\widehat{AB}$ is the arc bounding the sector. Another way to write the sector formula is $A = \frac{central\ angle}{360^\circ} \cdot \pi r^2$.

The last part of a circle that we can find the area of is called a segment, not to be confused with a line segment. A **segment of a circle** is the area of a circle that is bounded by a chord and the arc with the same endpoints as the chord. The **area** of a segment is $A_{segment} = A_{sector} - A_{\triangle ABC}$



Example A

Find the area of the blue sector. Leave your answer in terms of π.

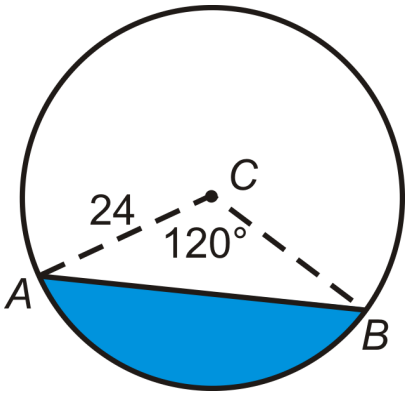


In the picture, the central angle that corresponds with the sector is 60°. 60° would be $\frac{1}{6}$ of 360°, so this sector is $\frac{1}{6}$ of the total area.

$$area\ of\ blue\ sector = \frac{1}{6} \cdot \pi 8^2 = \frac{32}{3}\pi$$

Example B

Find the area of the blue segment below.



As you can see from the picture, the area of the segment is the area of the sector minus the area of the isosceles triangle made by the radii. If we split the isosceles triangle in half, we see that each half is a 30-60-90 triangle, where the radius is the hypotenuse. Therefore, the height of $\triangle ABC$ is 12 and the base would be $2\left(12\sqrt{3}\right) = 24\sqrt{3}$.

$$A_{sector} = \frac{120}{360}\pi \cdot 24^2 \qquad A_{\triangle} = \frac{1}{2}\left(24\sqrt{3}\right)(12)$$
$$= 192\pi \qquad = 144\sqrt{3}$$

The area of the segment is $A = 192\pi - 144\sqrt{3} \approx 353.8$.

Example C

The area of a sector of circle is 50π and its arc length is 5π. Find the radius of the circle.

First substitute what you know to both the sector formula and the arc length formula. In both equations we will call the central angle, "CA."

$$50\pi = \frac{CA}{360}\pi r^2 \qquad 5\pi = \frac{CA}{360}2\pi r$$
$$50 \cdot 360 = CA \cdot r^2 \qquad 5 \cdot 180 = CA \cdot r$$
$$18000 = CA \cdot r^2 \qquad 900 = CA \cdot r$$

Now, we can use substitution to solve for either the central angle or the radius. Because the problem is asking for the radius we should solve the second equation for the central angle and substitute that into the first equation for the central angle. Then, we can solve for the radius. Solving the second equation for CA, we have: $CA = \frac{900}{r}$. Plug this into the first equation.

$$18000 = \frac{900}{r} \cdot r^2$$
$$18000 = 900r$$
$$r = 20$$

Watch this video for help with the Examples above.

MEDIA

Click image to the left for more content.

CK-12 Foundation: Chapter10AreaofSectorsandSegmentsB

Concept Problem Revisited

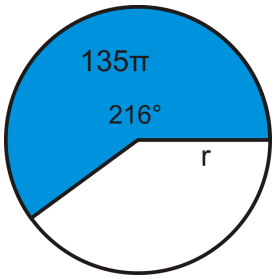
The area of the crust for a deep-dish pizza is $8^2\pi - 7^2\pi = 15\pi$. The area of the crust of the thin crust pizza is $8^2\pi - 7.5^2\pi = \frac{31}{4}\pi$. One-twelfth of the deep dish pizza has $\frac{15}{12}\pi$ or $\frac{5}{4}\pi$ in^2 of crust. One-fourth of the thin crust pizza has $\frac{31}{16}\pi$ in^2. To compare the two measurements, it might be easier to put them both into decimals. $\frac{5}{4}\pi \approx 3.93$ in^2 and $\frac{31}{16}\pi \approx 6.09$ in^2. From this, we see that one-fourth of the thin-crust pizza has more crust than one-twelfth of the deep dish pizza.

Vocabulary

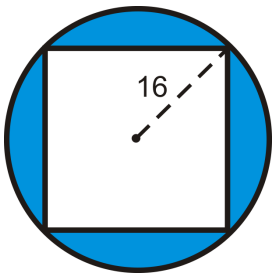
A ***circle*** is the set of all points that are the same distance away from a specific point, called the ***center***. A ***radius*** is the distance from the center to the outer rim of the circle. A ***chord*** is a line segment whose endpoints are on a circle. A ***diameter*** is a chord that passes through the center of the circle. The length of a diameter is two times the length of a radius. ***Area*** is the amount of space inside a figure and is measured in square units. π, or ***"pi"*** is the ratio of the circumference of a circle to its diameter. A ***sector of a circle*** is the area bounded by two radii and the arc between the endpoints of the radii. A ***segment of a circle*** is the area of a circle that is bounded by a chord and the arc with the same endpoints as the chord.

Guided Practice

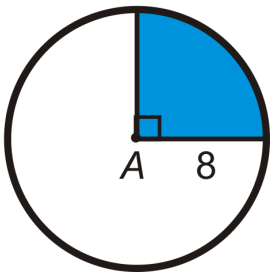
1. The area of a sector is 135π and the arc measure is $216°$. What is the radius of the circle?



2. Find the area of the shaded region. The quadrilateral is a square.



3. Find the area of the blue sector of $\odot A$.



Answers:

1. Plug in what you know to the sector area formula and solve for r.

$$135\pi = \frac{216^\circ}{360^\circ} \cdot \pi r^2$$
$$135 = \frac{3}{5} \cdot r^2$$
$$\frac{5}{3} \cdot 135 = r^2$$
$$225 = r^2 \rightarrow r = \sqrt{225} = 15$$

2. The radius of the circle is 16, which is also half of the diagonal of the square. So, the diagonal is 32 and the sides would be $\frac{32}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = 16\sqrt{2}$ because each half of a square is a 45-45-90 triangle.

$$A_{circle} = 16^2\pi = 256\pi$$
$$A_{square} = \left(16\sqrt{2}\right)^2 = 256 \cdot 2 = 512$$

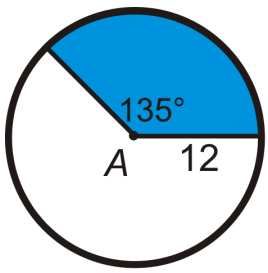
The area of the shaded region is $256\pi - 512 \approx 292.25$

3. The right angle tells us that this sector represents $\frac{1}{4}$ of the circle. The area of the whole circle is $A = \pi 8^2 = 64\pi$. So, the area of the sector is $\frac{1}{4}64\pi = 16\pi$.

Practice

Find the area of the blue sector or segment in $\odot A$. Leave your answers in terms of π. You may use decimals or fractions in your answers, but do not round.

1.



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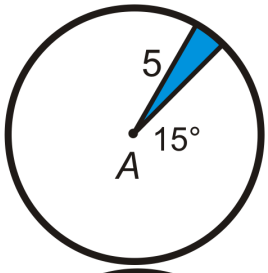

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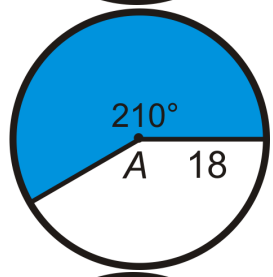

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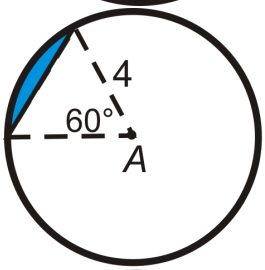

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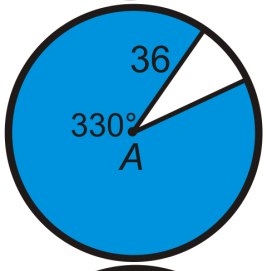

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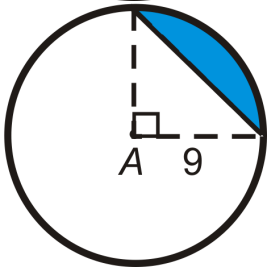
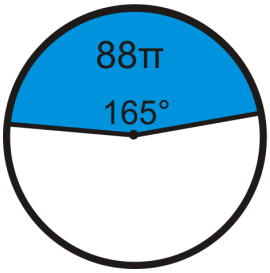


Find the radius of the circle. Leave your answer in simplest radical form.

7.



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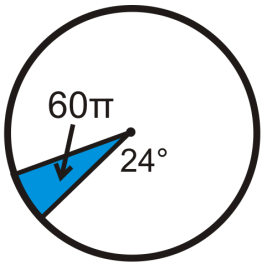

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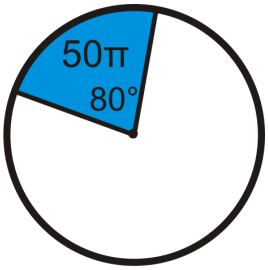

Find the central angle of each blue sector. Round any decimal answers to the nearest tenth.

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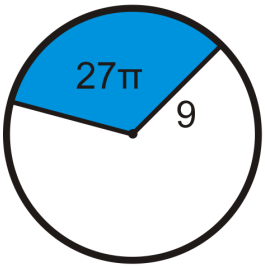

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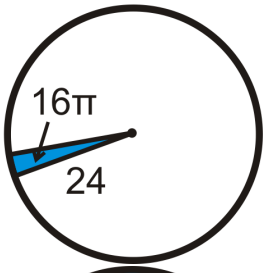

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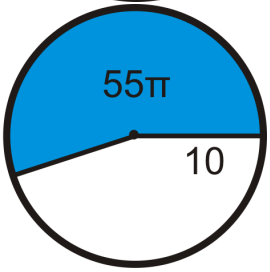


13. The area of a sector of a circle is 54π and its arc length is 6π. Find the radius of the circle.
14. Find the central angle of the sector from #13.
15. The area of a sector of a circle is 2304π and its arc length is 32π. Find the central angle of the sector.

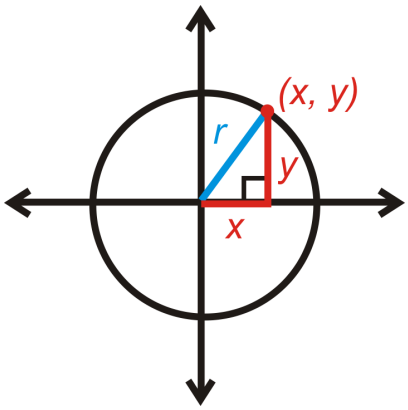
3.11 Writing and Graphing the Equations of Circles

Learning Objectives

- Graph a circle.
- Find the equation of a circle in the coordinate plane.
- Find the radius and center, given the equation of a circle and vice versa.
- Find the radius and center by completing the square.
- Find the equation of a circle, given the center and a point on the circle.
- Given the center and a point on the circle, prove or disprove an additional point is on the circle.

Graphing a Circle in the Coordinate Plane

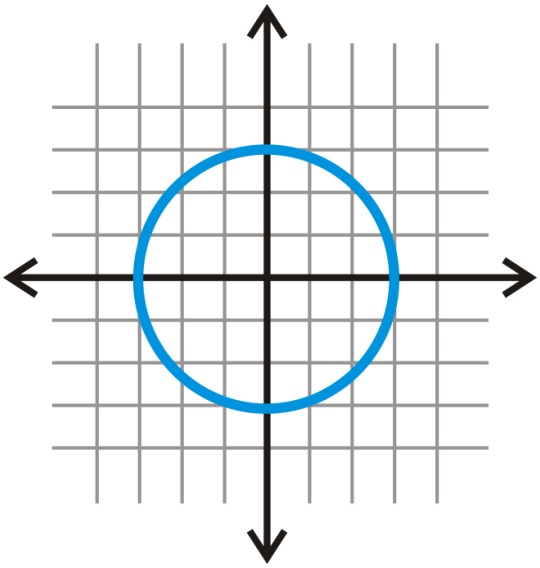
Recall that the definition of a circle is the set of all points that are the same distance from a point, called the center. This definition can be used to find an equation of a circle in the coordinate plane.



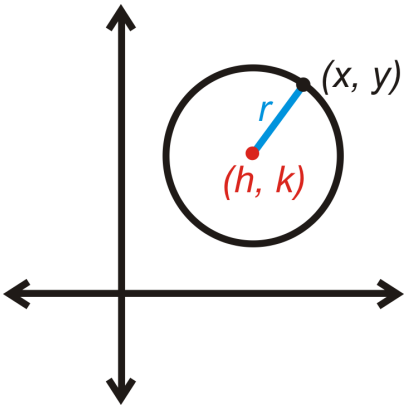
Let's start with the circle centered at the origin, (0, 0). If (x,y) is a point on the circle, then the distance from the center to this point would be the radius, r. x is the horizontal distance of the coordinate and y is the vertical distance. Drawing those in, we form a right triangle. Therefore, the equation of a circle, *centered at the origin* is $x^2+y^2=r^2$, by the Pythagorean Theorem.

Example 1: Graph $x^2+y^2=9$.

Solution: This circle is centered at the origin. It's radius is the square root of 9, or 3. The easiest way to graph a circle is to plot the center, and then go out 3 units in every direction and connect them to form a circle.



The center does not always have to be on (0, 0). If it is not, then we label the center (h,k) and would use the distance formula to find the length of the radius.



$$r = \sqrt{(x-h)^2+(y-k)^2}$$

If you square both sides of this equation, then we would have the standard equation of a circle.

Standard Equation of a Circle: The standard equation of a circle with center (h,k) and radius r is $r^2 = (x-h)^2 + (y-k)^2$.

Example 2: Find the center and radius of the following circles.

a) $(x-3)^2+(y-1)^2 = 25$

b) $(x+2)^2+(y-5)^2 = 49$

Solution:

a) Rewrite the equation as $(x-3)^2+(y-1)^2 = 5^2$. Therefore, the center is (3, 1) and the radius is 5.

b) Rewrite the equation as $(x-(-2))^2+(y-5)^2 = 7^2$. From this, the center is (-2, 5) and the radius is 7.

When finding the center of a circle always take the ***opposite sign*** of what the value is in the equation.

Example 3: What are the center and radius of the circle described by the equation $x^2+y^2-12x+8y+27 = 0$

In order to find the center and radius, you will need to complete the square. For a video that highlights this concept, click here.

$x^2+y^2-12x+8y=-27$ Begin by moving the constant to the right side of the equation.

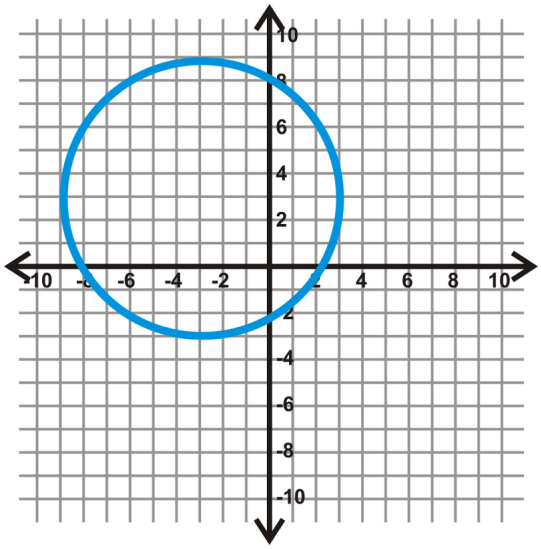
$x^2+y^2-12x+36+8y+16=-27+36+16$ To complete each square, you will take half of the coefficient of x and square it (which is where the 36 came from) and half of the coefficient of y and square it (which is where the 16 came from). Remember that what you add to one side of the equation, you must add to the other to keep the balance of the equation.

$x^2-12x+36+y^2+8y+16=25$ When we organize our equation into like terms and simplify, we can factor into perfect squares.

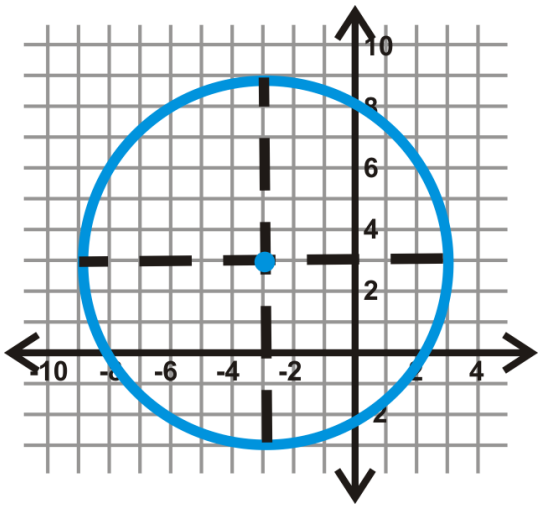
$(x-6)^2+(y+4)^2=25$

Completing the square and factoring allows us to identify the center at (6, -4) and a radius of 5.

Example 4: Find the equation of the circle below.



Solution: First locate the center. Draw in a couple diameters. It is easiest to use the horizontal and vertical diameters.



From the intersecting diameters, we see that the center is (-3, 3). If we count the units from the center to the circle on either of these diameters, we find that the radius is 6. Plugging this information into the equation of a circle, we get $(x-(-3))^2+(y-3)^2=6^2$ or $(x+3)^2+(y-3)^2=36$.

Finding the Equation of a Circle

Example 5: Find the equation of the circle with center (4, -1) and passes through (-1, 2).

Solution: To find the equation, first plug in the center to the standard equation.

$(x-4)^2+(y-(-1))^2=r^2$ or $(x-4)^2+(y+1)^2=r^2$

Now, plug in (-1, 2) for x and y and solve for r.

$$\begin{aligned}(-1-4)^2+(2+1)^2&=r^2\\(-5)^2+(3)^2&=r^2\\25+9&=r^2\\34&=r^2\end{aligned}$$

At this point, we don't need to solve for r because r^2 is what is in the equation. Substituting in 34 for r^2, we have $(x-4)^2+(y+1)^2=34$.

Example 6:

Prove or disprove that the point $(1, \sqrt{3})$ lies on the circle centered at the origin and containing the point $(0,2)$.

We know that the circle has a center of $(0,0)$ and contains the point $(0,2)$. Graphing these points will help us see the radius of the circle.

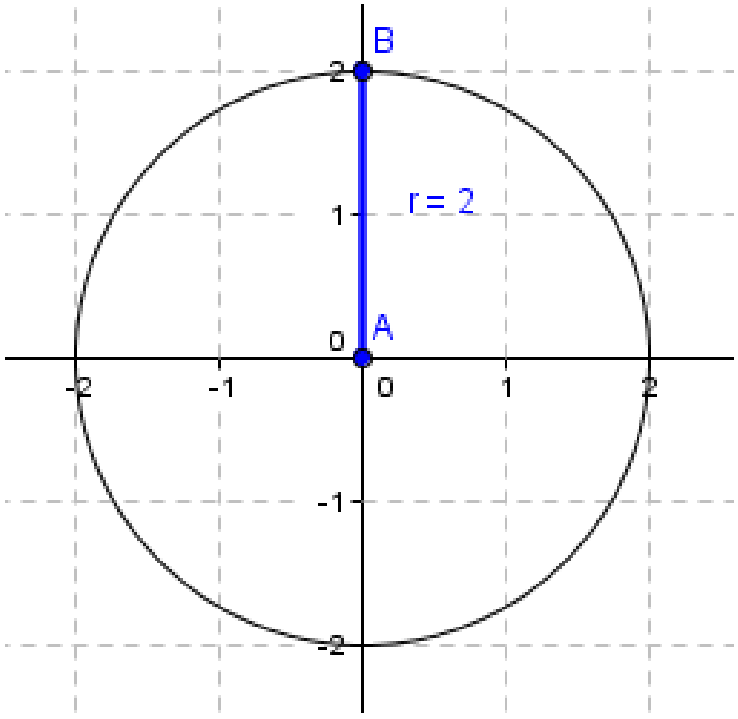


FIGURE 3.4

With a center of $(0,0)$ and a radius of 2, we know the equation of our circle is $x^2+y^2=4$. To prove or disprove that the point $(1, \sqrt{3})$ is on the circle, we plug the values in for x and y to see if they make the equation true.

$(1)^2 + (\sqrt{3})^2 = 4$
$1 + 3 = 4$
$4 = 4$

Since the values make the equation true, the point $(1, \sqrt{3})$ is on the circle. You can verify this by using your graphing calculator or geometric software such as Geogebra (note that point C is estimating $\sqrt{3}$):

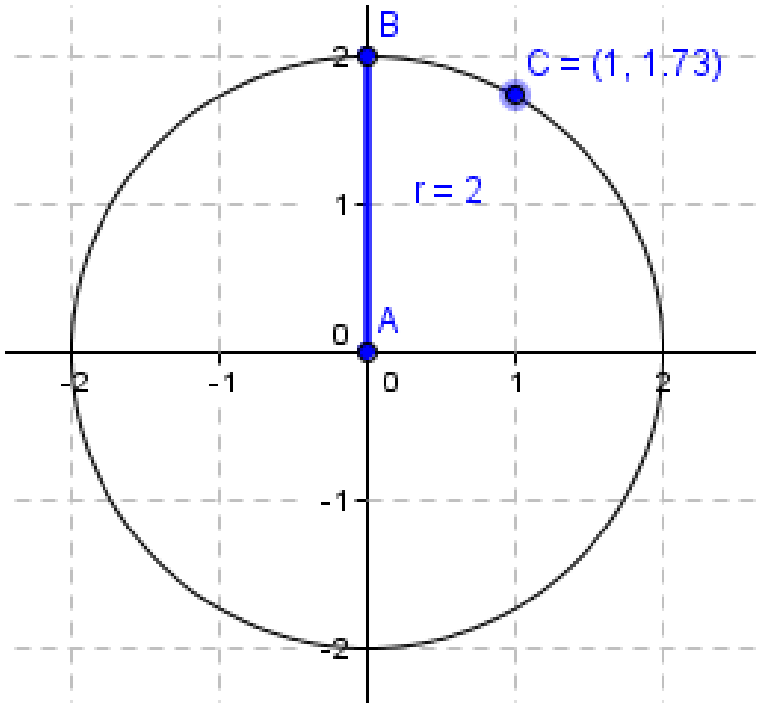


FIGURE 3.5

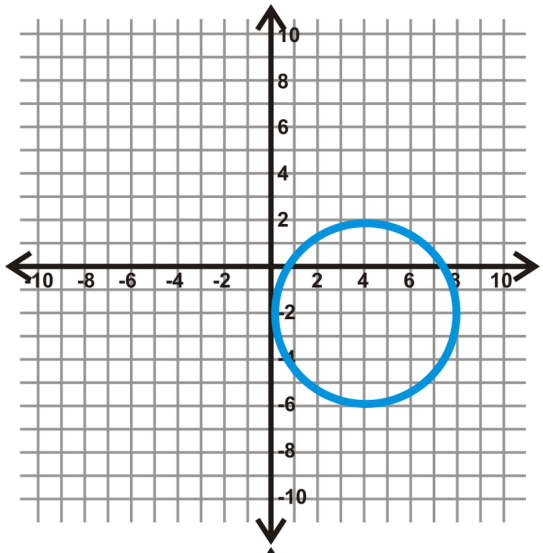
Review Questions

Find the center and radius of each circle. Then, graph each circle.

1. $(x+5)^2 + (y-3)^2 = 16$
2. $x^2 + (y+8)^2 = 4$
3. $(x-7)^2 + (y-10)^2 = 20$
4. $(x+2)^2 + y^2 = 8$

Find the equation of the circles below.

5.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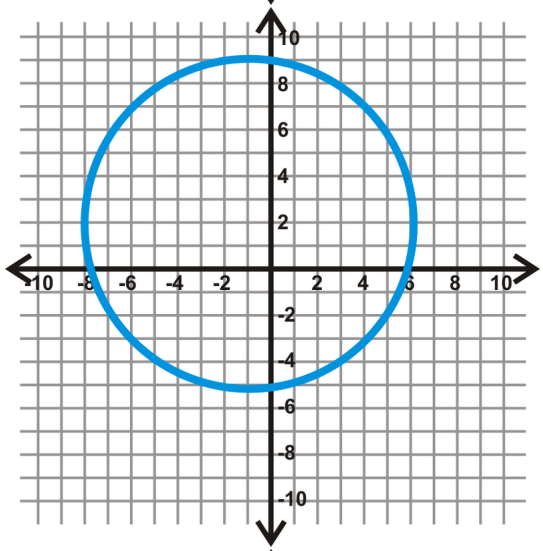
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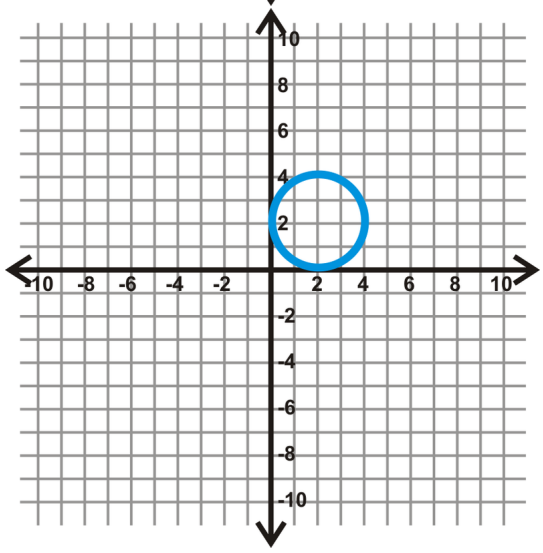
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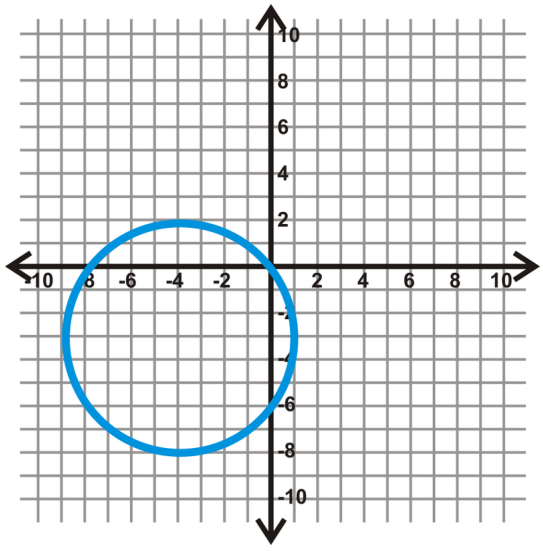
8. 

9. Determine if the following points are on $(x+1)^2+(y-6)^2=45$.
   a. (2, 0)
   b. (-3, 4)
   c. (-7, 3)

Find the equation of the circle with the given center and point on the circle.

10. center: (2, 3), point: (-4, -1)
11. center: (10, 0), point: (5, 2)
12. center: (-3, 8), point: (7, -2)
13. center: (6, -6), point: (-9, 4)
14. Now let's find the equation of a circle using three points on the circle. Do you remember how we found the center and radius of a circle given three points on the circle in problem 30 of Section 9-3? We used the fact that the perpendicular bisector of any chord in the circle will pass through the center. By finding the perpendicular bisectors of two different chords and their intersection we can find the center of the circle. Then we can use the distance formula with the center and a point on the circle to find the radius. Finally, we will write the equation. Given the points $A(-12,-21), B(2,27)$ and $C(19,10)$ on the circle (an arc could be drawn through these points from A to C), follow the steps below.
    a. Since the perpendicular bisector passes through the midpoint of a segment we must first find the midpoint between A and B.
    b. Now the perpendicular line must have a slope that is the opposite reciprocal of the slope of $\overleftrightarrow{AB}$. Find the slope of $\overleftrightarrow{AB}$ and then its opposite reciprocal.
    c. Finally, you can write the equation of the perpendicular bisector of $\overline{AB}$ using the point you found in part a and the slope you found in part b.
    d. Repeat steps a-c for chord $\overline{BC}$.
    e. Now that we have the two perpendicular bisectors of the chord we can find their intersection. Solve the system of linear equations to find the center of the circle.
    f. Find the radius of the circle by finding the distance from the center (point found in part *e*) to any of the three given points on the circle.
    g. Now, use the center and radius to write the equation of the circle.

Find the equations of the circles which contain three points in problems 15 and 16.

15. $A(-2,5), B(5,6)$ and $C(6,-1)$
16. $A(-11,-14), B(5,16)$ and $C(12,9)$

17. What are the center and radius of the circle described by the equation $x^2 + y^2 + 14x - 4y + 44 = 0$?
18. What are the center and radius of the circle described by the equation $x^2 + y^2 - 6x + 4y + 9 = 0$?
19. What are the center and radius of the circle described by the equation $x^2 + y^2 - 10x - 8y + 25 = 0$?
20. Prove or disprove that the point $(2, \sqrt{5})$ lies on the circle centered at the origin and containing the point $(0, 3)$.
21. Prove or disprove that the point $(-1, -2)$ lies on the circle centered at $(2, -2)$ and containing the point $(2, 1)$.