Calculus Concept Collection - Chapter 2

Introduction to Limits

Answers

1. See the vocabulary section above.

2. a. $\lim_{x\to -6} f(x) = \infty$ means $f(x)$ has a vertical asymptote at x=-6, i.e., it gets larger without bound as x approaches -6;
   b. $\lim_{x\to \infty} f(x) = -6$ has a horizontal asymptote at y=-6, i.e., $f(x)$ is bounded by y=-6 as x gets larger.

3. $\lim_{x\to \infty} f(x) = 200$ means that $f(x)$ has a horizontal asymptote at y=200, i.e., $f(x)$ approaches 200 as x get larger.

4. $\lim_{x\to 175} f(x) = 175$ means that as x approaches 175 from the left or the right, $f(x)$ approaches 175.

Evaluate the following limits, if they exist. If a limit does not exist, explain why.

5. $\lim_{t\to\infty} \frac{3t^2-7t}{t-8} = \lim_{t\to\infty} \frac{t^2(3-\frac{7}{t})}{t(1-\frac{8}{t})} = \lim_{t\to\infty} 3t = \infty$

6. $\lim_{t\to\infty} 3 = 3$

7. $\lim_{t\to\infty}(t^2-t^4) = \lim_{t\to\infty} t^4(\frac{1}{t^2}-1) = -\infty$

8. $\lim_{x\to\infty} x+\sqrt{x^2+2x} = \infty$

9. $h(g) = \frac{5g^2-7g+9}{g^2-2g-3} = \frac{5g^2-7g+9}{(g+1)(g-3)}$, with vertical asymptotes at g=-1 and g=3;

   $\lim_{x\to\pm\infty} h(g) = \lim_{x\to\pm\infty} \frac{g^2(5-\frac{7}{g}+\frac{9}{g^2})}{g^2(1-\frac{2}{g}-\frac{3}{g^2})} = 5$, means a horizontal asymptote at $\pm\infty$.

Given: $f(x) = \frac{x^2-x-6}{x^2-2x-8}$ perform the following:

10. $f(x)=\dfrac{x^2-x-6}{x^2-2x-8}=\dfrac{(x-3)(x+2)}{(x-4)(x+2)}=\dfrac{(x-3)}{(x-4)}$ has a hole at (-2, $\frac{5}{6}$), and a vertical asymptote at x=4.

$\lim\limits_{x\to\pm\infty} f(x)=\lim\limits_{x\to\pm\infty}\dfrac{(x-3)}{(x-4)}=1$, means a horizontal asymptote at y=1.

Since $\lim\limits_{x\to 4^-}\dfrac{(x-3)}{(x-4)}=-\infty$ and $\lim\limits_{x\to 4^+}\dfrac{(x-3)}{(x-4)}=\infty$, $\lim\limits_{x\to 4}\dfrac{(x-3)}{(x-4)}$ is not defined.

11. $f(x)=\dfrac{x^2-x-6}{x^2-2x-8}$ has a root at x=3, y-intercept at (0, $\frac{3}{4}$), and hole at (-2, $\frac{5}{6}$).

12. For $n>0$, $\lim\limits_{t\to\infty}\dfrac{1}{t^n}=0$

13. For $n<0$, $\lim\limits_{t\to\infty}\dfrac{1}{t^n}=\infty$

14. For $n=0$, $\lim\limits_{t\to\infty}\dfrac{1}{t^0}=\lim\limits_{t\to\infty}\dfrac{1}{1}=1$

15. If the degree of G is less than the degree of H, $\lim\limits_{x\to\infty}\dfrac{G(x)}{H(x)}=0$.

16. If the degree of G is greater than the degree of H, $\lim\limits_{x\to\infty}\dfrac{G(x)}{H(x)}$ is either ∞ or $-\infty$.

17. If the degree of G is the same as the degree of H, $\lim\limits_{x\to\infty}\dfrac{G(x)}{H(x)}$ is a constant.

18. a. The amount of salt (in grams) after t minutes is $30\square 25t$. The volume of water (in liters) after t minutes is $8000+25t$. Therefore the concentration, $C(t)$, of salt after t minutes is

$$C(t)=\frac{30\square 25t}{8000+25t}$$.

b. $\lim\limits_{t\to\infty}C(t)=\lim\limits_{t\to\infty}\dfrac{30\square 25t}{8000+25t}=\lim\limits_{t\to\infty}\dfrac{25t\square 30}{25t(\frac{320}{t}+1)}=30$ g/L. This makes sense since the longer the pool fills, the more the concentration of salt in the pool looks like the additive.

One-sided Limits

Answers

1. $\lim\limits_{x\to -3^-} = 5$
2. $\lim\limits_{x\to 2^+} = -3$
3. $\lim\limits_{x\to -1^+} = -8$ and $\lim\limits_{x\to -1^-} = 2$.
4. $\lim\limits_{x\to -1} = 2$
5. $\lim\limits_{x\to -2^-} = -2$ and $\lim\limits_{x\to 5^+} = 5$

6. $\lim\limits_{x\to 2^+} \dfrac{-x^2 - 2x + 8}{x-2} = \lim\limits_{x\to 2^+} \dfrac{-(x+4)(x-2)}{x-2} = -6$

7. $\lim\limits_{x\to -3} g(x) = 1$

8. $\lim\limits_{x\to 0^+} \dfrac{-x^2 + 4x}{x} = \lim\limits_{x\to 0^+} (-x+4) = 4$

9. $\lim\limits_{x\to -1} g(x) = -5$

10. $\lim\limits_{x\to 1^+} \dfrac{4x^2 - x - 3}{x-1} = \lim\limits_{x\to 1^+} \dfrac{(4x+3)(x-1)}{x-1} = 7$

11. $\lim\limits_{x\to 3} f(x) = 4$

12. $\lim\limits_{x\to 0^+} \dfrac{x^2 - 4x}{x} = \lim\limits_{x\to 0^+} (x-4) = -4$

13. $\lim\limits_{x\to 2} h(x) = 12$

14. $\lim\limits_{x\to 2^-} \dfrac{4x^2 - 7x - 2}{x-2} = \lim\limits_{x\to 2^-} \dfrac{(4x+1)(x-2)}{x-2} = 9$

15. $\lim\limits_{x\to -2} g(x) = -7$

16. $\lim_{x \to 3} g(x) = -9$

17. $\lim_{x \to -5^-} \frac{-3x^2 - 13x + 10}{x+5} = \lim_{x \to -5^-} \frac{-(3x-2)(x+5)}{x+5} = 17$

18. $\lim_{x \to 2} f(x) = 3$

Properties of Limits

Answers

1. Use the addition rule: $\lim_{x\to -2}(3x^2+4x-9)=\lim_{x\to -2}3x^2+\lim_{x\to -2}4x-\lim_{x\to -2}9=-5$

2. Use the product rule. As $x\to 0$:

$$\lim 5^x=1,\ \lim\cos(\pi x)=1, \text{ so then } \lim(5^x\cdot\cos(\pi x))=1\cdot 1=1.$$

3. Use the addition rule. As $x\to 1$:

$$\lim 5^x=5,\ \lim\cos(\pi x)=0, \text{ so then } \lim(5^x+\cos(\pi x))=5+0=5.$$

4. No. The limit will not exist if $g(x)$ does not have a limit.

5. $\lim_{x\to 3}\dfrac{x^2+7x+12}{x+3}=\dfrac{\lim_{x\to 3}x^2+7x+12}{\lim_{x\to 3}x+3}=\dfrac{42}{6}=7$

6. $\lim_{x\to 2}\sqrt[3]{x^3+4x^2+3}=\sqrt[3]{\lim_{x\to 2}(x^3+4x^2+3)}=\sqrt[3]{27}=3$

7. $\lim_{x\to -2}(x+4)^3(2x-1)^2=\left(\lim_{x\to -2}[x+4]\right)^3(\lim_{x\to -2}[2x-1])^2=8\square 25=200$

8. $\lim_{x\to -7}\dfrac{x^2+7x+12}{x+7}$ does not exist because the denominator is 0 at $x=-7$.

9. Yes. Since both functions are continuous, their limits are equal to their value. The product of two continuous functions is continuous, so its value is equal to its limit as well. By the product rule, the limit of the product is the product of the limits.

10. By definition, $g(f(x))=f(x)f(x)+f(x)$. Now, we know that the limit of $f(x)$ as $x\to 0$ is five. Apply the product rule and the addition rule to find that:

$$\lim g(f(x))=5^2+5=25+5=30$$

11. Anything raised to the zero power is one, and cosine of zero is one. Therefore use the limit rules to find that the limit is equal to:

$$5e^0+2\pi^0+\frac{0+2}{\cos(0)}=5+2+2=5+4=9$$

12. e^π . Plugging pi into the formula and using the limit rules gives us:

$$\frac{3\sin(\pi)}{\pi^9+\pi^5+10\pi^4+9\pi+1}+e^\pi(\cos(\sin(\pi+2\pi)))=\frac{0}{\pi^9+\pi^5+10\pi^4+9\pi+1}+e^\pi(\cos(\sin(3\pi)))$$

$$=e^\pi(\cos(0))=e^\pi$$

13. By the limit rules, the limit of $f(x)$ must be a root of the polynomial $p(x) = x^3 + x^2 + x + 1$. We have to factor the polynomial: $p(x) = x^3 + x^2 + x + 1 = (x+1)(x^2+1)$ The only root is at one, so the limit of $f(x)$ as $x \to 0$ is 1.

14. You cannot take the limit as x approaches zero because this would be dividing by zero.

15. Use the product rule.

$$\lim f(x)^5 = \lim f(x) f(x) f(x) f(x) f(x) = \lim f(x) \lim f(x) \lim f(x) \lim f(x) \lim f(x)$$

$= (\lim f(x))^5$. This method is not enough to prove the product rule because it will not prove the case where the function is raised to a non-integer power.

Limits Involving Infinity

Answers

1) The degree of the numerator is greater than the degree of the denominator, so the function will grow without bound. Since the denominator is *x - 7*, the function cannot include *x = 7*, because the function cannot be defined where the denominator = *0*. Logically, as *x* gets huge, the *-7* matters less and less, and we end up with just x^2.

2) Similar to the last problem, the numerator is of greater degree than the denominator, so the function does not approach a limit. The denominator is *x - 3*, so the graph cannot include *3*.

3) As $x \to \infty$, or in other words, "as *x* gets huge", the value of x^3 grows even faster, either positive or negative, so there is no limit.

4) As *x* grows huge, x^2 grows much faster than the rest of the expression, therefore, we can approximate the end behavior of $2x^2 + 3x - 7$ with $y = x^2$

5) $-2x^3 - 5x^2 + 8x$ is a 3rd degree equation, so it will turn twice, since it is not a rational function, there are no concerns about numerator or denominator. The function will have no limits, and will grow without bound in both the positive and negative directions. If you use a graphing calculator to graph the function, you will see that $y = x^3$ can be used to approximate it.

6) $$\lim_{x \to 3+} \frac{(x+2)^2}{(x-2)^2 - 1} = +\infty$$

7) $$\lim_{x \to \infty} \frac{(x+2)^2}{(x-2)^2 - 1} = 1$$

8) $$\lim_{x \to 1+} \frac{(x+2)^2}{(x-2)^2 - 1} = -\infty$$

9) $$\lim_{x \to \infty} \frac{2x-1}{x+1} = 2$$

10) $\lim_{x \to -\infty} \frac{x^5 + 3x^4 + 1}{x^3 - 1} = -\infty$

11) $\lim_{x \to \infty} \frac{3x^4 - 2x^2 + 3x + 1}{2x^4 - 2x^2 + x - 3} = \frac{3}{2}$

12) $\lim_{x \to \infty} \frac{2x^2 - x + 3}{x^5 - 2x^3 + 2x - 3} = 0$

13) Zero at $x = -4$; vertical asymptotes at $x = 3, 5$; $f(x) \to 1$ as $x \to \pm\infty$.

14) Zero at $x = 1$; no vertical asymptotes; $f(x) \to -\infty$ as $x \to \infty$; $f(x) \to \infty$ as $x \to -\infty$.

15) Zero at $x = 2$; no vertical asymptotes but there is a discontinuity at $x = -2$; $f(x) \to -\infty$ as $x \to -\infty$; $f(x) \to \infty$ as $x \to \infty$.

Limits of Polynomial and Rational Functions

Answers

Solve the following rational function limits.

1. $\lim_{x\to 1}\frac{-12x^2+12}{4x-4}=\lim_{x\to 1}[-3(x+1)]=-6$

2. $\lim_{x\to 2}\frac{\frac{3}{x+3}-\frac{11}{9}}{2x-4}=\lim_{x\to 2}\frac{-11x-6}{18(x+3)(x-2)}=\text{Does } Not\ Exist$ (One-sided limits do not agree):

   $\lim_{x\to 2^-}\frac{\frac{3}{x+3}-\frac{11}{9}}{2x-4}=\infty$ and $\lim_{x\to 2^+}\frac{\frac{3}{x+3}-\frac{11}{9}}{2x-4}=-\infty$

3. $\lim_{x\to\frac{-57}{56}}\frac{\frac{-5x-3}{2x+3}-\frac{13}{6}}{-56x-57}=\lim_{x\to\frac{-57}{56}}\frac{1}{12x+18}=0.173$

4. $\lim_{x\to 2}\frac{2x^2-5x+2}{-x+2}=\lim_{x\to 2}-(2x-1)=-3$

5. $\lim_{x\to\frac{3}{4}}\frac{4x^2+5x-6}{4x-3}=\lim_{x\to\frac{3}{4}}(x+2)=\frac{11}{4}$

6. $\lim_{x\to 4}\frac{\frac{-3}{-2x+3}-\frac{3}{5}}{6x-24}=\lim_{x\to 4}\frac{-6x+25}{30(2x-3)(x-4)}=DNE$

   $\lim_{x\to 4^-}\frac{\frac{-3}{-2x+3}-\frac{3}{5}}{6x-24}=-\infty$ and $\lim_{x\to 4^+}\frac{\frac{-3}{-2x+3}-\frac{3}{5}}{6x-24}=\infty$.

7. $\lim_{x\to\frac{-3}{2}}\frac{\frac{-4x-3}{-2x+2}-\frac{5}{2}}{-2x-3}=\lim_{x\to\frac{-3}{2}}\frac{x-8}{2(2x+3)(x-1)}=DNE$

$$\lim_{x\to\frac{-3}{2}^-}\frac{\frac{-4x-3}{-2x+2}-\frac{5}{2}}{-2x-3}=-\infty$$ and $$\lim_{x\to\frac{-3}{2}^+}\frac{\frac{-4x-3}{-2x+2}-\frac{5}{2}}{-2x-3}=\infty$$.

8. $\lim_{x\to 4}\frac{x^2-8x+16}{x-4}=\lim_{x\to 4}(x-4)=0$

9. $\lim_{x\to\frac{10}{39}}\frac{\frac{3x+3}{-3x+4}-\frac{7}{6}}{39x-10}=\lim_{x\to\frac{10}{39}}\frac{-1}{6(3x-4)}=0.052$

10. $\lim_{x\to -4}\frac{3x^2+7x-20}{-x-4}=\lim_{x\to -4}-(3x-5)=17$

11. $\lim_{x\to -4}\frac{4x^2+14x-8}{x+4}=\lim_{x\to -4}(4x-2)=-18$

12. $\lim_{x\to\frac{-18}{13}}\frac{\frac{-4x+1}{-3x+5}-\frac{5}{7}}{-13x-18}=\lim_{x\to\frac{-18}{13}}\frac{-1}{7(3x-5)}=0.016$

13. $\lim_{x\to -2}\frac{\frac{3}{x+4}-\frac{3}{2}}{-3x-6}=\lim_{x\to -2}\frac{-3x-6}{-3x-6}=1$

14. $\lim_{x\to\frac{-3}{4}}\frac{\frac{-x+3}{4x+3}-\frac{11}{2}}{-4x-3}=\lim_{x\to\frac{-3}{4}}\frac{46x+27}{2(4x+3)^2}=-\infty$

15. $\lim_{x\to\frac{1}{4}}\frac{-8x^2-2x+1}{-4x+1}=\lim_{x\to\frac{1}{4}}(2x+1)=\frac{3}{2}$

16. $\lim_{x\to\frac{1}{4}}\frac{16x^2-16x+3}{-4x+1}=\lim_{x\to\frac{1}{4}}-(4x-3)=2$

17. $\lim_{x\to 0}\frac{x^2+3x}{x}=\lim_{x\to 0}(x+3)=3$

Limits Involving Radical Functions

Answers

1. $\lim_{x\to 3}\sqrt{x}=\sqrt{3}$

2. $\lim_{x\to 8}\sqrt{x-7}=1$

3. $\lim_{x\to 4}\frac{\sqrt{x}-2}{x-4}=\frac{1}{4}$

4. $\lim_{x\to 1}\frac{\sqrt{x+3}-2}{x-1}=\frac{1}{4}$

5. $\lim_{x\to 0^+}\frac{\sqrt{x}}{\sqrt{1+\sqrt{x}}-1}=2$

6. $\lim_{x\to\infty}\left(\sqrt{x^2-5x}-x\right)=-\frac{5}{2}$

7. $\lim_{x\to\infty}\frac{\sqrt{x^6+3x^2+1}}{4x^3+3}=\frac{1}{4}$

8. $\lim_{x\to 0}\frac{\sqrt{x+5}-\sqrt{5}}{x}=\frac{\sqrt{5}}{10}$

9. $\lim_{x\to 3}(\sqrt{x^2+4x})=\sqrt{21}$

10. $\lim_{x\to -1}(x^2+2x+10)^{3/2}=27$

11. $\lim_{x\to 4}\frac{\sqrt{x+5}-3}{x-4}=\frac{1}{6}$

12. $\lim_{x\to 1}(\sqrt{2x^3+3x^2+7})=2\sqrt{3}$

13. $\lim_{x\to 3}(\sqrt[3]{2x^2-10})=2$

14. $\lim_{x\to 7}\frac{5x}{\sqrt{x+2}}=\frac{35}{3}$

15. $\lim_{x\to 0}\frac{7-\sqrt{x^2+49}}{x}=0$

Limits Involving Trigonometric Functions

Answers

1. $\lim_{x\to\pi/6}\sin(x)=\frac{1}{2}$

2. $\lim_{x\to\pi/4}\cot(x)=1$

3. $\lim_{x\to\pi/3}\sec^2(x)=4$

4. $\lim_{x\to\pi/3}[\sin(x)+\cos(x)]=\frac{1+\sqrt{3}}{2}$

5. $\lim_{x\to\pi/2}[\sec(x)\tan(x)]=Does\ Not\ Exist$

6. $\lim_{x\to\pi}\frac{\sin x}{\cos x+1}=Does\ Not\ Exist$

7. The constant can be pulled out of the limit:

$$\lim_{x\to 0}\frac{\sin(x)}{3x}=\frac{1}{3}\lim_{x\to 0}\frac{\sin(x)}{x}=\frac{1}{3}\times 1=\frac{1}{3}.$$

8. 2 can be factored from the numerator and pulled out of the limit:

$$\lim_{x\to 0}\frac{2\cos(x)-2}{x}=\lim_{x\to 0}\frac{2(\cos(x)-1)}{x}=2\lim_{x\to 0}\frac{\cos(x)-1}{x}=2\times 0=0.$$

9. To evaluate this limit, we need a 3 in the denominator to match the one inside the sine function. To introduce one, we must multiply both the numerator and denominator by 3:

$$\lim_{x\to 0}\frac{\sin(3x)}{x}=\lim_{x\to 0}\frac{3\sin(3x)}{3x}$$

And remove the 3 in the numerator from the limit.

$$\lim_{x\to 0}\frac{3\sin(3x)}{3x}=3\lim_{x\to 0}\frac{\sin(3x)}{3x}=3\times 1=3.$$

10. Let $y=\frac{1}{x}$. Then as $x\to\infty, y\to 0$. The limit becomes

$$\lim_{x\to\infty}x\sin(\frac{1}{x})=\lim_{y\to 0}\frac{1}{y}\sin(y)=\lim_{y\to 0}\frac{\sin(y)}{y}=1$$

11. $\lim_{x\to 1}\frac{\cos(x^2-1)-1}{x^2-1}=0$ (Use $u=x^2-1$ substitution and $u\to 0$ in the limit)

12. Since the numerator approaches 1 as the denominator approaches zero, this expression will not approach a finite value. Specifically,

$$\lim_{x\to 0^-}\frac{\cos(x)}{x}=\frac{1}{0^-}=-\infty$$

$$\lim_{x\to 0^+}\frac{\cos(x)}{x}=\frac{1}{0^+}=\infty$$

So the limit does not exist.

13. $\lim\limits_{x\to 0}(2x\cot 2x)=\lim\limits_{x\to 0}\left(\frac{2x}{\sin 2x}\cos 2x\right)=1$.

14. $\lim\limits_{x\to -3/2}\left[\frac{\sin(2x+3)}{2x^2+x-3}\right]=\lim\limits_{x\to -3/2}\left[\frac{\sin(2x+3)}{(2x+3)(x-1)}\right]=\lim\limits_{x\to -3/2}\left[\frac{\sin(2x+3)}{2x+3}\,\frac{1}{x-1}\right]=(1)\cdot(-\frac{2}{5})=-\frac{2}{5}$

15. $\lim\limits_{x\to 0}\left[\frac{\tan x}{x}\right]=\lim\limits_{x\to 0}\left[\frac{\sin x}{x}\cdot\frac{1}{\cos x}\right]=[1\cdot 1]=1$

Limits of Composite Functions

Answers:

1. $f(x) = x^2 + 3 = g \circ h$, where $g(x) = x + 3$ and $h(x) = x^2$.

2. $f(x) = \sqrt{x^2 + 4x} = g \circ h$, where $g(x) = \sqrt{x}$ and $h(x) = x^2 + 4x$.

3. $f(x) = (x^2 + 2x + 10)^{3/2} = g \circ h$, where $g(x) = x^{3/2}$ and $h(x) = x^2 + 2x + 10$.

4. $f(x) = \sin(x^2 + 3) = g \circ h$, where $g(x) = \sin x$ and $h(x) = x^2 + 3$.

5. $f(x) = 2^{\sin x} = g \circ h$, where $g(x) = 2^x$ and $h(x) = \sin x$.

6. $\lim_{x \to 2}(x^2 + 3) = \lim_{x \to 2}(x^2) + 3 = 7$

7. $\lim_{x \to 3}(\sqrt{x^2 + 4x}) = \sqrt{\lim_{x \to 3}(x^2 + 4x)} = \sqrt{21}$

8. $\lim_{x \to -1}(x^2 + 2x + 10)^{3/2} = [\lim_{x \to -1}(x^2 + 2x + 10)]^{3/2} = [9]^{3/2} = 27$

9. $\lim_{x \to 1}(\sqrt{2x^3 + 3x^2 + 7}) = \sqrt{\lim_{x \to 1}(2x^3 + 3x^2 + 7)} = 2\sqrt{3}$

10. $\lim_{x \to \sqrt{\pi/2}} \sin(x^2 + \pi) = \sin[\lim_{x \to \sqrt{\pi/2}}(x^2 + \pi)] = -1$

11. $\lim_{x \to \pi/2} 2^{\sin x} = 2^{\lim_{x \to \pi/2}(\sin x)} = 2^1 = 2$

12. $\lim_{x \to -1} \sin(2^x \frac{\pi}{2}) = \sin[\lim_{x \to -1}(2^x \frac{\pi}{2})] = \sin[\frac{\pi}{4}] = \frac{\sqrt{2}}{2}$

13. $\lim_{x \to 3}(\sqrt[3]{2x^2 - 10}) = \sqrt[3]{\lim_{x \to 3}(2x^2 - 10)} = \sqrt[3]{8} = 2$

14. $\lim_{x \to -1} e^{2t^2} = e^{\lim_{x \to -1}(2t^2)} = e^2$

15. $\lim_{x \to -1} \frac{x^3 + 1}{x + 1} = \lim_{x \to -1} \frac{(x+1)(x^2 - x + 1)}{x + 1} = \lim_{x \to -1}(x^2 - x + 1) = 3$

Continuity of a Function

Answers

1. While graph of the function appears to be continuous everywhere, a check of the table values indicates that the function is not continuous at $x = -1$.

2. While the function appears to be continuous for all $x = -2$, a check of the table values indicates that the function is not continuous at $x = 2$.

3. Test for continuity by evaluating the limit from either side, then evaluating the function.

$$\lim_{x\to 3^-} |x-3| = \lim_{x\to 3^-}(-(x-3)) = \lim_{x\to 3^-} -x+3 = -(3)+3 = 0$$

$$\lim_{x\to 3^+} |x-3| = \lim_{x\to 3^+} x-3 = 3-3 = 0$$

$$f(3) = |3-3| = |0| = 0$$

Because $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) = f(3)$, the function is continuous at the point.

4. Test for continuity by evaluating the limit from either side, then evaluating the function.

$$f(x) = \frac{6x-1}{4x^2+20x+25} = \frac{6x+1}{(2x+5)^2}$$

$$\lim_{x\to -5/2^-} f(x) = \lim_{x\to -5/2^+} f(x) = -\infty$$

$f(-\frac{5}{2})$ does not exist. Therefore the function is not continuous.

5. Test for continuity by evaluating the limit from either side, then evaluating the function.

$$f(x) = \frac{3x^2-13x-10}{x^3-4x^2-2x-15} = \frac{(3x+2)(x-5)}{(x-5)(x^2+x+3)}$$

$$\lim_{x\to 5^-} f(x) = \lim_{x\to 5^+} f(x) = \frac{17}{33}$$

$f(5)$ does not exist.

The function is not continuous at the point.

6. $k = -2$

7.

(a) $\lim_{x\to -3^-} f(x) = \lim_{x\to -3^-} \sqrt{1-x} = \sqrt{1-(-3)} = \sqrt{4} = 2$

$$\lim_{x\to -3^+} f(x) = \lim_{x\to -3^+} \frac{4}{3}x+6 = \frac{4}{3}(-3)+6 = -4+6 = 2$$

$f(-3)$ does not exist, but since the limits agree with one another, $x=-3$ is a removable discontinuity.

(b) $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} \frac{4}{3}x+6 = \frac{4}{3}(3)+6 = 4+6 = 10$

$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} x^2 = 3^2 = 9$

$f(3)=10$

Even though we can evaluate the function at $x=3$, the limits do not agree with each other so $x=3$ is a jump discontinuity.

8. For both parts (a) and (b), it will be useful to have the function in factored form.

$$f(x) = \frac{3x^3+x^2-27x-9}{3x^2+2x-1} = \frac{x^2(3x-1)-9(3x-1)}{(3x-1)(x+1)} = \frac{(x^2-9)(3x-1)}{(3x-1)(x+1)} = \frac{(x+3)(x-3)(3x-1)}{(3x-1)(x+1)}$$

(a) At $x=-1$ the function's numerator approaches a finite value while the denominator approaches 0. This is a pole type discontinuity.

(b) Observe that $f(x) = \frac{(x+3)(x-3)(3x-1)}{(3x-1)(x+1)}$ is equal to $\frac{(x+3)(x-3)}{(x+1)}$ everywhere that they both exist; however, $f(x)$ does not exist at $x=\frac{1}{3}$. Therefore

$$\lim_{x\to\frac{1}{3}} f(x) = \lim_{x\to\frac{1}{3}} \frac{(x+3)(x-3)}{(x+1)} = \frac{-182}{3}.$$

Since the limit exists, though the function does not, the discontinuity is removable.

9. The function will be discontinuous only where the denominator equals zero. We find these points by setting each term in the product equal to zero and find $x=0$ and $x=-\frac{1}{4}$. $x^2+1=0$ has no solution. The fact that $4x+1$ also appears in the denominator does not mean there is no discontinuity due to this term; it only means that it is a removable singularity instead of an asymptote. Since the function is defined everywhere except $x=0$ and $x=-\frac{1}{4}$, The intervals on which x is continuous are $(-\infty,-\frac{1}{4})$, $(-\frac{1}{4},0)$ and $(0,\infty)$.

10. Because the sum of continuous functions is continuous, we look at the continuity of each term separately. 3 has no discontinuities. $\frac{2}{x-3}$ is discontinuous only at 3. Therefore the intervals of continuity of $f(x)$ are $(-\infty,3)$ and $(3,\infty)$.

11. $\cot(x)$ is undefined when $\tan(x)=0$. $\cot^2(x)$ is the same. $\tan(x)=0$ at $0, \pi, 2\pi, 3\pi...$ any integer multiple of π. Therefore $f(x)$ is undefined at these points, and the intervals of continuity of $f(x)$ are $...(-2\pi,-\pi),(-\pi,0),(0,\pi),(\pi,2\pi),(2\pi,3\pi),...$ There are infinitely many.

12. False. $f(x)$ is continuous everywhere its denominator is nonzero. But, the denominator is zero at $x=\frac{1}{2}$ which is in the interval [0,1]. $f(x)$ is not continuous on this interval.

13. False. The denominator of $f(x)$ is zero at $\pi/3$ which is in the interval $[\pi/4, \pi/2]$. Since the function is undefined at $\pi/3$ it is not continuous there.

14. $f(x)=\sqrt{x^2-4x+2}$ is undefined, therefore not continuous, in the open interval $(2-\sqrt{2})<x<(2+\sqrt{2})$ because the integrand of the function is <0 in that interval. It is continuous everywhere else.

15. $f(x)=\sqrt{x^2+4x+5}$ is defined everywhere and therefore continuous everywhere.

Properties of Continuous Functions

Answers

1. f(-3)=-5 and f(-2)=3, so there is at least one root in the interval

2. f(9)<0 and f(10)>0, so there is at least one root in the interval.

3. f(-3)>0 and f(0)<0, so there is at least one root in the interval.

4. f(-0.5)>0 and f(0)<0, so there is at least one root in the interval.

5. f(-2.75)>0 and f(0)<0, so there is at least one root in the interval.

6. True. First observe that this function is continuous everywhere, so it is continuous on the interval $(0,\pi)$. Next, test the endpoints of the interval. $f(0)=\sin(0)+\cos(0)=0+1=1$, $f(\pi)=\sin(\pi)+\cos(\pi)=0-1=-1$. So the IVT guarantees a root on the interval $(0,\pi)$.
7. False. The IVT can only guarantee that certain intervals will contain roots; it says nothing about which intervals will not contain roots.
8. False. Although we can confirm that $f(0)>0$ and $f(\pi)<0$, The function is discontinuous where the denominator, $\cos(x)=0$. This occurs at $x=\frac{\pi}{2}$, which is on the interval $(0,\pi)$. Therefore, the IVT does not apply to this problem. Furthermore, we can guarantee that this function has no roots anywhere on the real line since its numerator is never equal to zero.

9. We can confirm that this function is continuous everywhere, so it is continuous on all of the above intervals. Next we check the values of $f(x)$ at the endpoints. $f(0)>0,\ f(\frac{2\pi}{3})>0$ so the IVT makes no guarantee about the interval $(0,\frac{2\pi}{3})$. $f(-\frac{\pi}{3})>0,\ f(\frac{\pi}{3})>0$ so, again, the IVT says nothing about the interval in option (b).

    $f(-\frac{2\pi}{3})<0$, however, and we already know $f(0)>0$. So there is a root on the interval $(-\frac{2\pi}{3},0)$. Since $(-\frac{2\pi}{3},0)$ is contained in $(-\pi,\pi)$, we know there is also a root on this interval without examining its endpoints. The correct answers are (c) and (d).
10. Answers will vary. First confirm that this function is continuous on any given interval. Next, note that $e^x>0$ everywhere, so $f(x)>0$ for any $x\geq 0$. Thus $x=0$ is a good right endpoint for our interval. It remains to find a left endpoint with $f(x)<0$. $x=-1$ is a fine choice; here

$f(-1) = e^{-1} - 1 = \frac{1}{e} - 1 \approx \frac{1}{2.72} - 1 < 0$. So one interval is (-1,0). Most negative numbers work for the left endpoint, and any positive number for the right endpoint, as well.

11. Intervals that include the function's roots at $x = -5, -2, 2$ are correct.

12. False. The function is not continuous on the interval [-3, 3].

13. True. The function is continuous on the interval (1, 3).

14. True. The function is continuous on the interval (-3, 0).

15. False. The function is not continuous on the interval [-3, 0].