**Calculus Concept Collection - Chapter 4**

# Related Rates

## Answers

1. Answers will vary.
2. $\frac{dy}{dt} = -\frac{3}{2}\frac{\text{ft}}{\text{sec}}$
3. Using the following diagram, $\frac{dy}{dt} = \frac{720}{\sqrt{5200}}\frac{\text{ft}}{\text{sec}} \approx 10.18\frac{\text{ft}}{\text{sec}}.$

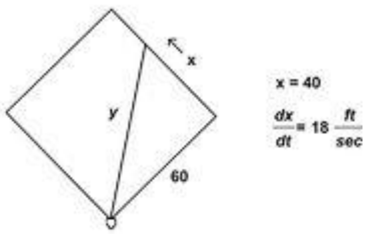


**Figure 4.1.5**

4. Using the following diagram, $\frac{dy}{dt} = \frac{6000}{\sqrt{100000}}\frac{\text{ft}}{\text{sec}} \approx 18.97\frac{\text{ft}}{\text{sec}}.$

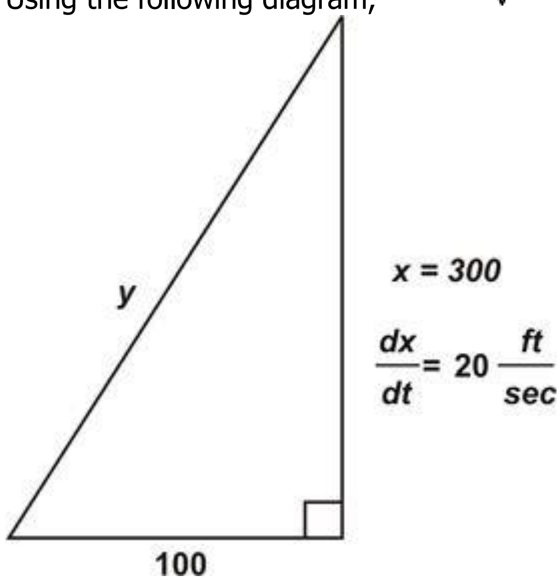


**Figure 4.1.6**

5. Using the following diagram, $\frac{ds}{dt} = \frac{17550}{\sqrt{31300}} \approx 99.20 \text{ mph.}$

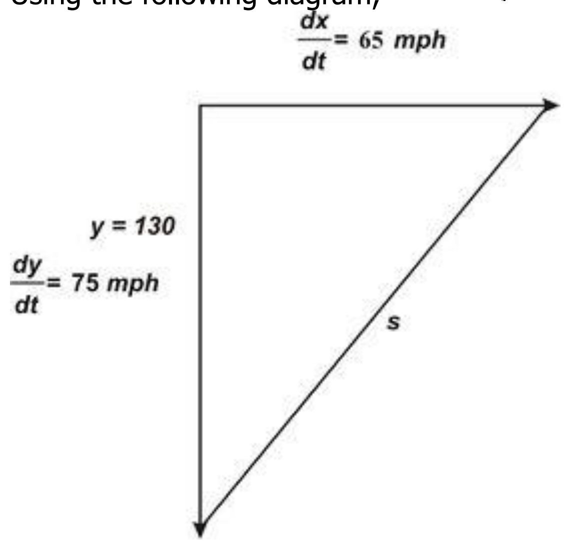


**Figure 4.1.7**

6. Using the following diagram, $\frac{dx}{dt} = -\frac{16}{5}\frac{\text{ft}}{\text{sec}}.$

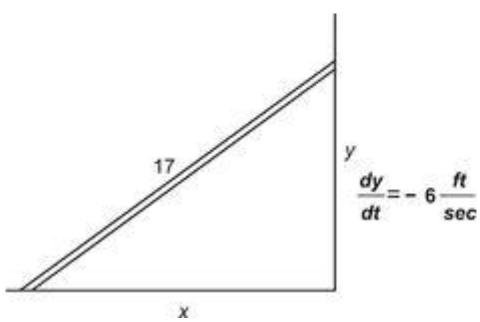


**Figure 4.1.8**

7. $\frac{dA}{dt} = 140\frac{\text{ft}}{\text{min}}$

8. The demand is increasing at a rate of $1/4$ per thousand units, or $250 \text{ units}$ per week.

9. $\frac{dV}{dt} = 27\frac{\text{in}^3}{\text{min}}$

10.

    1. $\frac{dr}{dt} = \frac{2}{\pi}\frac{\text{in}}{\text{min}}$
    2. $\frac{dp}{dt} = 4\frac{\text{in}}{\text{min}}$

11. $100\pi$ square centimeters per second.

The area of circle is given by the formula $A = \pi r^2$, so then the change in the area is given by $\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$. Plugging in the values from the problem we have that $\frac{dA}{dt} = 2\pi(10)(5) = 100\pi$.

12. $\frac{1}{\pi}$.

The area of circle is given by the formula $A = \pi r^2$, so then the change in the area is given by $\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$. Plugging in the values from the problem we have that $\frac{dA}{dt} = 100 = 2\pi r(50) = 100\pi r$, so then $r = \frac{100}{100\pi} = \frac{1}{\pi}$.

13. $\frac{5}{3}$ centimeters a second.

$V = \pi r^2 h$ for a cylinder, which implies that $\frac{dV}{dt} = \pi r^2 \frac{dH}{dt}$ when radius is held constant. Plugging in the value from the problem into the last formula gives $10 = 6\frac{dH}{dt}$, so that $\frac{dH}{dt} = \frac{10}{6} = \frac{5}{3}$.

14. $V = \pi r^2 h$ for a cylinder, which implies that $\frac{dV}{dt} = 2\pi r \frac{dr}{dt} h$.

15. We do not need to fully understand the situation in the problem in order to come to the right answer.

Since the variables other than $x_1$ are fixed, we can take them not to be variables at all and instead just view them as constants. Taking the derivative of the momentum function, then, gives us:

$$\frac{dM}{dt} = \pi\cos(\pi x_1)\frac{dx_1}{dt} + 5x_1^4\ln(\frac{5}{x_4}x_4)\frac{dx_1}{dt} = -8,000,000\times(\pi\cos(\pi x_1) + 5x_1^4\ln(5))$$

Now, all we have to do is plug in the value $x_1 = 1$.

$$\frac{dM}{dt} = -8,000,000\times(\pi\cos(\pi) + 5\ln(5)) = -8,000,000\times(5\ln(5) - \pi).$$

This is our final answer.

# Finding Maxima and Minima (Extrema) of Functions

## Answers

1. Absolute max at $x = 7$, absolute minimum at $x = 4$, relative maximum at $x = 2$.
   Note: there is no relative minimum at $x = 9$ because there is no open interval around $x = 9$. Since the function is defined only on $x = 9$ the extreme values of $f$ are $f(7) = 7$, $f(4) = 0$.

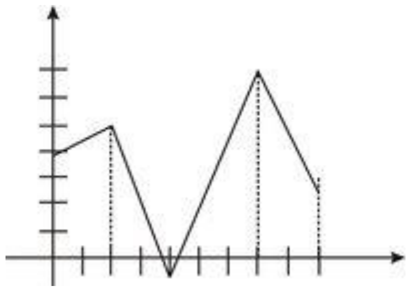

**Figure 4.2.8**

2. Absolute maximum at $x = 7$, absolute minimum at $x = 9$, relative minimum at $x = 3$,
   Note: there is no relative minimum at $x = 0$ because there is no open interval around $x = 0$ since the function is defined only on $[0, 9]$; the extreme values of $f$ are
   $f(7) = 9, f(9) = 0.$

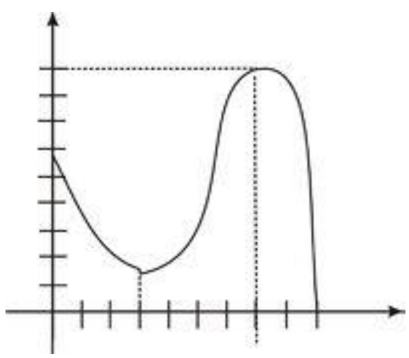

**Figure 4.2.9**

3. Absolute minimum at $x = 0, f(0) = 1$; there is no maximum since the function is not continuous on a closed interval.

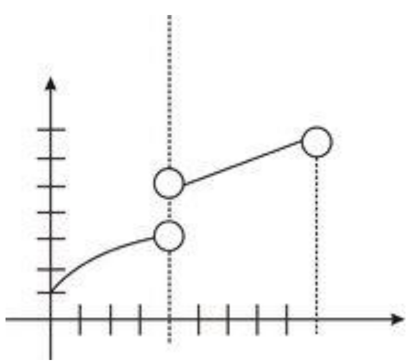

**Figure 4.2.10**

4. Absolute maximum at $x=-3,\quad f(-3)=13,$ absolute minimum at $x=1,\quad f(1)=-3$

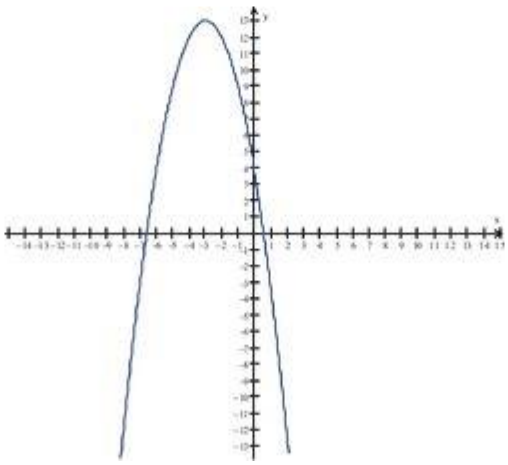

**Figure 4.2.11**

5. Absolute maximum at $x=\frac{3}{4},\ f(\frac{3}{4})\approx .1055,$ absolute minimum at $x=2,\quad f(2)=-8.$

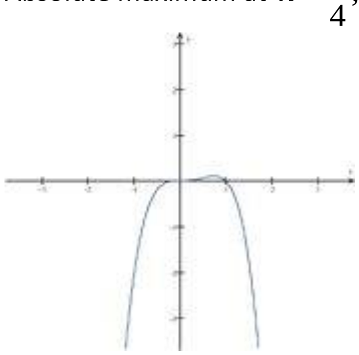

**Figure 4.2.12**

6. Absolute minimum at $x=-\sqrt{2},\ f(-\sqrt{2})=4.$

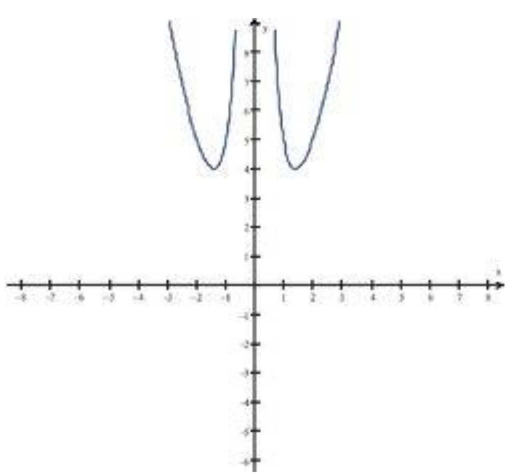

**Figure 4.2.13**

7. The sine function is bounded above by one everywhere on the real line, so if it achieves a value of one at any point in an interval then that must be the absolute maximum for that interval. We know that $\sin(\frac{\pi}{2}) = 1$, so the absolute maximum is one.
8. The function can achieve local minimums at the end points of the interval or at the critical points. Since the derivative of cosine, sine, is zero at only at the end points and pi, those are the only three points we have to check. Both end points are absolute maximums, so the only minimum is at pi, where the function equals negative one. This is also the absolute minimum.

9. The derivative of $x^2$ is $2x$, which is always positive over the interval [6,7], so the function is always increasing. Since this is the case, the absolute maximum and minimum must occur at the end points of the interval. Therefore the minimum value of the function is 36 and the maximum is 49.

10. The absolute minimum is zero, which occurs at $x = 8$.
    The function involves a quadratic. Usually when this is the case it is a good idea to try and factor the quadratic; in this case we can rewrite the function as
    $f(x) = (x^2 - 7x - 8)^4 = ((x-8)(x+1))^4$. We then have that the function equals zero only at the points eight and negative one. Since the function is a fourth power, it can never be negative, so if an interval contains either the point eight or negative one then the absolute minimum occurs at that point and is zero. Our interval contain eight, so this is the case.

11. It must then be true that $f(x)$ has a slope of zero, and achieves both its maximum and its minimum at the same time everywhere. In other words, the function is equal to a constant.

12. First observe that since it is a polynomial, the domain of this function is all real numbers.
    To find critical numbers, begin by taking a derivative.
    $f'(x) = 12x^3 - 8x^2 - 12x + 8$
    The critical numbers are *x*-values in the domain of the function at which the derivative is equal to zero or at which the derivative does not exist. Since the derivative is still a polynomial, its domain is still all real numbers, so there are none of the latter type of critical numbers. Find where the derivative equals zero.
    $0 = 12x^3 - 8x^2 - 12x + 8$
    $0 = 3x^3 - 2x^2 - 3x + 2$
    $0 = x^2(3x-2) - 1(3x-2)$
    $0 = (x^2 - 1)(3x-2)$
    $0 = (x+1)(x-1)(3x-2)$

The critical numbers are $x = -1$, $x = 1$ and $x = \frac{2}{3}$.

13. Critical numbers are *x*-values in the domain of the function at which the derivative is equal to zero or at which the derivative does not exist. Note that the function's domain does not include *x*-values such that $2x^2 - 5x - 3 = 0$, that is, $x = -\frac{1}{2}$ and $x = 3$ are excluded from the domain.

Now find the derivative.

$$f(x) = \frac{1}{2x^2 - 5x - 3} = (2x^2 - 5x - 3)^{-1}$$

$$f'(x) = -1 \times (2x^2 - 5x - 3)^{-2} \times \frac{d}{dx}(2x^2 - 5x - 3) = \frac{-1}{(2x^2 - 5x - 3)^2} \times (4x - 5) = -\frac{4x - 5}{(2x^2 - 5x - 3)^2}$$

The values at which the derivative does not exist are just $x = -\frac{1}{2}$ and $x = 3$ again. Since they are outside the function's domain, they cannot be critical numbers of the function. To see where the derivative equals zero, set the numerator equal to zero.

$$4x - 5 = 0 \rightarrow x = \frac{5}{4}$$

$x = \frac{5}{4}$ is the only critical number of the function.

14. First find the domain of the function. Numbers that are not positive are outside the domain of the logarithm function, so we require $x - 4 > 0 \rightarrow x > 4$ as the domain of the function itself. Next take a derivative.

$$f'(x) = \frac{d}{dx}\ln(x - 4) = \frac{1}{x - 4} \times \frac{d}{dx}(x - 4) = \frac{1}{x - 4}.$$

The derivative cannot be evaluated at *x*=4. However, 4 is outside the function's domain, so it cannot be a critical number of the function. Next we set the derivative equal to zero and solve. For a fraction to equal zero, its numerator must equal zero. However, $1 \neq 0$ so $f'(x)$ is never zero.

The function has no critical numbers.

15. First find the domain of the function. Since negative values are outside the domain of the square root function, the domain of $f(x)$ is $x \geq 0$. Now take the derivative.

$$f(x) = \sqrt{x} \times (1 - x) = \sqrt{x} + x\sqrt{x} = x^{1/2} - x^{3/2}$$

$$f'(x) = \frac{1}{2}x^{-1/2} - \frac{3}{2}x^{1/2} = \frac{1}{2\sqrt{x}} - \frac{3\sqrt{x}}{2} = \frac{1}{2\sqrt{x}} - \frac{3\sqrt{x} \times \sqrt{x}}{2\sqrt{x}} = \frac{1 - 3|x|}{2\sqrt{x}}$$

Find where the derivative cannot be evaluated. That is, where the denominator is zero or the quantity inside the square root is negative. The denominator is zero at *x*=0. *x*=0 is in the domain of the function, but not in the domain of the derivative, so it is one critical number. The quantity inside the square root is negative when $x<0$. These values are not in the function's domain, so they are not critical numbers. Finally, find values that make the derivative equal to zero by setting its numerator equal to zero.

$1-3|x|=0 \rightarrow |x|=\frac{1}{3} \rightarrow x=\pm\frac{1}{3}$. $x=-\frac{1}{3}$ is outside the domain of the function. $x=\frac{1}{3}$ is within the domain; therefore it is a critical number.

The critical numbers of *f* are $x=\frac{1}{3}$ and $x=0$.

16. First find the function's domain. The function exists for all real *x*-values except those which make the denominator zero, that is, $x=2$.
Now find the derivative.

$$f'(x)=\frac{(2x+1)\times(x-2)-1\times(x^2+x+1)}{(x-2)^2}=\frac{x^2-2x-3}{(x-2)^2}=\frac{(x-3)(x+1)}{(x-2)^2}$$

The derivative cannot be evaluated at the point $x=2$, however, that is outside the function's domain and is therefore not a critical number. Now find *x*-values for which the derivative is zero by setting the numerator equal to zero. These are $x=3$ and $x=-1$.
Therefore the critical numbers are $x=3$ and $x=-1$.

17. Yes, by the extreme value theorem. The function is continuous over a closed interval, therefore it achieves an absolute minimum there.

18. Yes. Even though the extreme value theorem does not guarantee a maximum, the absolute maximum of sine over the interval is $\sin(\frac{\pi}{2})=1$, making the absolute maximum of the function over the interval 7. The function does not achieve an absolute minimum over the interval however.

19. No. The extreme value theorem does not apply as even though our interval is closed, the function is not continuous throughout the interval. Specifically, there is a discontinuity when $x=0$. By choosing x values arbitrarily closed to zero we can make the function arbitrarily large, so the answer is no.

20. No. The function grows arbitrarily close to zero as x approaches zero but since zero is not in the interval, and since the function is always positive over the interval, it does not have an absolute minimum value.

21. Yes, by the extreme value theorem. The interval is closed and the function is continuous over the entire interval.

# The Mean Value Theorem

## Answers

1. $f(x)=3x^3-12x=0$ at $x=0,\pm 2$; $f'(x)=9x^2-12=0$ at $c=\pm\dfrac{2\sqrt{3}}{3}$

2. $f(x)=x^2-\dfrac{2}{x-1}=0$ at only one value , so Rolle's Theorem does not apply.

3. $f(x)=-2x^2-12x+5=0$ at $x=\dfrac{-6\pm\sqrt{46}}{2}$; $f'(x)=-4x-12$ at $c=-3$ .

4. $f(x)=|2x-3|=0$ at only one value, so Rolle's Theorem does not apply.

5. $f(x)=2\sin x+3\cos x=0$ at multiple locations where $\tan x=-\dfrac{3}{2}$;

   $f'(x)=2\cos x-3\sin x=0$ at multiple locations where $\tan x=\dfrac{2}{3}$ or $c\approx 0.588\pm n\pi$ .

6. $f(x)=x^4-2x^2=0$ at $x=0,\pm\sqrt{2}$; $f'(x)=4x^3-4x=0$ at $c=0,\pm 1$

7. $x^3+a_1x^2+a_2x=x(x^2+a_1x+a_2)=0$ has a positive root at $x=r=\dfrac{-a_1+\sqrt{a_1^{\,2}-4a_2}}{2}$

   provided $a_2<0$; $f'(x)=3x^2+2a_1x+a_2=0$ at $x=\dfrac{-a_1\pm\sqrt{a_1^{\,2}-3a_2}}{3}<r$ .

8. $f'(c)=\dfrac{f(b)-f(a)}{b-a}=-1=-\dfrac{c+2}{c^2}+\dfrac{1}{c}$ ; $c=\sqrt{2}$ in [1,2].

9. $f'(c)=\dfrac{f(b)-f(a)}{b-a}=2=-\dfrac{2}{c^2}$; no real solution for $c$ .

10. $f'(c)=\dfrac{f(b)-f(a)}{b-a}=-2=2c-5$ ; $c=\dfrac{3}{2}$ in [0,3].

11. $f'(c)=\dfrac{f(b)-f(a)}{b-a}=1=\dfrac{4}{c^2}$ ; $c=2$ in [1,4].

12. $f'(c)=\dfrac{f(b)-f(a)}{b-a}=13=3c^2-8$ ; $c=\sqrt{7}$ in [1,4].

13. $f'(c)=\dfrac{f(b)-f(a)}{b-a}=\dfrac{2}{\pi}=\cos c$ ; $c=0.88$ radians in [0, $\pi/2$ ].

14. $f'(c)=\dfrac{f(b)-f(a)}{b-a}=-\dfrac{2}{\pi}=-\sin c$ ; $c=0.69$ radians in [0, $\pi/2$ ].

15. $f'(c)=\dfrac{f(b)-f(a)}{b-a}=\dfrac{7}{3}=2^c\ln 2$ ; $c=1.75$ in [0,3].

## The First Derivative Test

Answers

1. Increasing on $(0,3)$,decreasing on $(3,6)$,constant on $(6,+\infty)$.
2. Increasing on $(-\infty,0)$and $(3,7)$,decreasing on $(0,3)$.
3. $f'(-3)>0$,$f'(1)<0$,$f'(3)=0$,$f'(4)>0$
4. Relative minimum at $x=-\sqrt[3]{0.5}$; increasing on $(-\sqrt[3]{0.5},0)$and $(0,+\infty)$, decreasing on $(-\infty,-\sqrt[3]{0.5})$.

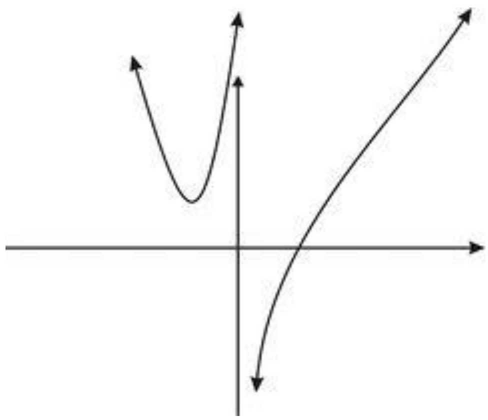

**Figure 4.4.8**

5. Absolute minimum at $x=0$; decreasing on $(-\infty,0)$, increasing on $(0,+\infty)$.

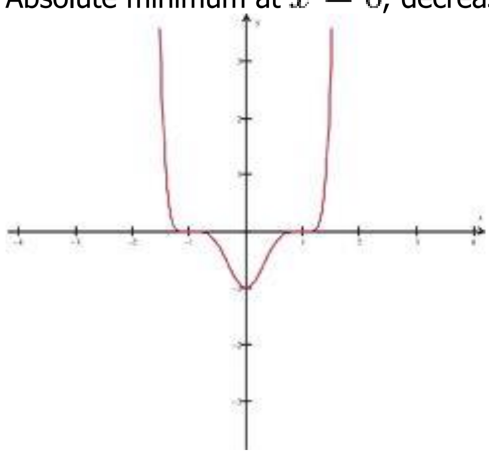

**Figure 4.4.9**

6. Absolute minimum at $x = \pm 1$; relative maximum at $x = 0$; decreasing on $(-\infty, -1), (0, 1)$, increasing on $(-1, 0)$, $(1, +\infty)$.

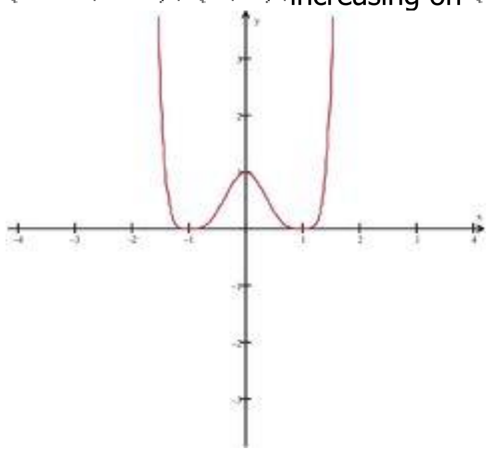

**Figure 4.4.10**

7. Absolute maximum at $x = -2$; increasing on $(-\infty, -2)$, decreasing on $(-2, +\infty)$.

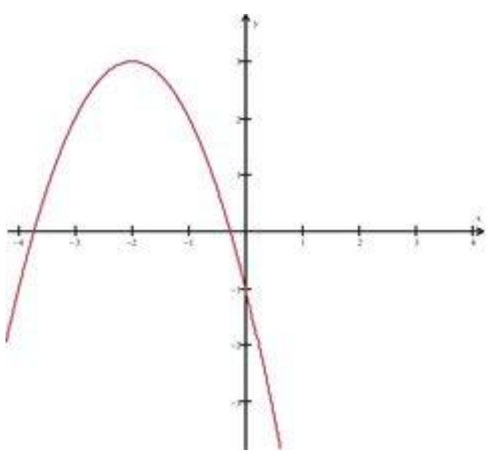

**Figure 4.4.11**

8. Relative maximum at $x=-3, f(-3)=28$; relative minimum at $x=1$, $f(1)=-4$; increasing on $(-\infty,-3)$ and $(1,+\infty)$, decreasing on $(-3,1)$.

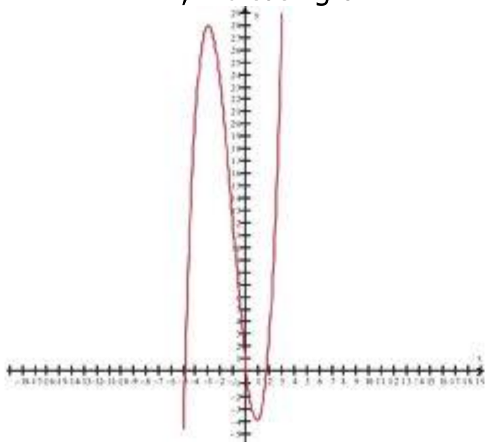

**Figure 4.4.12**

9. Relative maximum at $x=0, f(0)=0$; relative minimum at $x=2, f(2)=-3 \cdot 2^{\frac{2}{3}}=-3\sqrt[3]{4}, x=1, f(1)=-4$; increasing on

$(-\infty,0)$ and $(2,+\infty)$ decreasing on $(0,2)$.

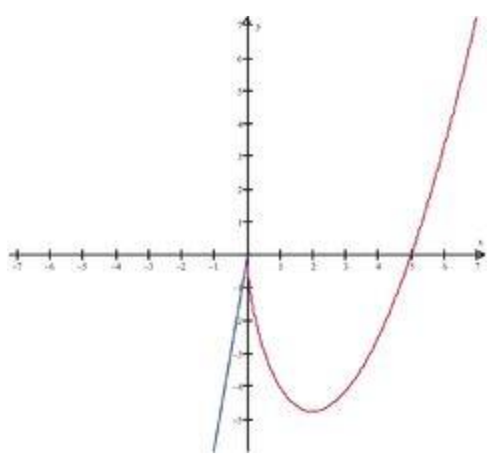

**Figure 4.4.13**

10. There are no maximums or minimums; no relative maximums or minimums.

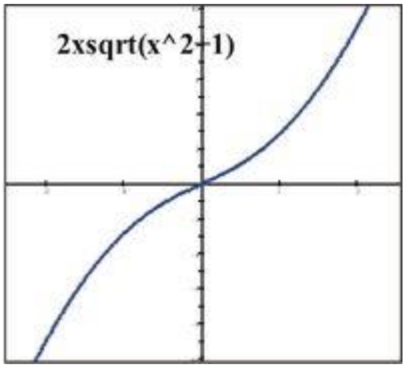


**Figure 4.4.14**

11. Although we are given critical numbers, we need to find all critical numbers of *f* in order to test values between them. Take the derivative of *f*.

$f(x) = \cos^2(x) = (\cos(x))^2$

$f'(x) = 2(\cos(x)) \times (-\sin(x)) = -2\cos(x)\sin(x)$

$f'(x) = 0$ where $\cos(x) = 0$ or $\sin(x) = 0$, that is,

$x = -2\pi, -\frac{3\pi}{2}, -\pi, -\frac{\pi}{2}, 0, \frac{\pi}{2}, \pi,, \frac{3\pi}{2}, 2\pi....$

And there are no other critical numbers since the derivative is continuous.

To classify the critical number at $x = 0$, choose a point on the interval $(-\frac{\pi}{2}, 0)$ and one on the interval $(0, \frac{\pi}{2})$. For example, we choose $x = -\frac{\pi}{4}, x = \frac{\pi}{4}$.

It is not necessary to calculate the value of the derivative exactly; just determine if it is positive or negative.

$f'(-\frac{\pi}{4}) = -2\sin(-\frac{\pi}{4})\cos(-\frac{\pi}{4}) = -2 \times -\frac{\sqrt{2}}{2} \times \frac{\sqrt{2}}{2} > 0$

$f'(\frac{\pi}{4}) = -2\sin(\frac{\pi}{4})\cos(\frac{\pi}{4}) = -2 \times \frac{\sqrt{2}}{2} \times \frac{\sqrt{2}}{2} < 0$

The function is increasing before *x*=0 and decreasing after it so *x*=0 is a local maximum.

To classify the critical number at $x = \frac{3\pi}{2}$, test values on the intervals $(\pi, \frac{3\pi}{2})$ and $(\frac{3\pi}{2}, 2\pi)$, for example, $x = \frac{5\pi}{4}, x = \frac{7\pi}{4}$.

$f'(\frac{5\pi}{4}) = -2\sin(\frac{5\pi}{4})\cos(\frac{5\pi}{4}) = -2\times -\frac{\sqrt{2}}{2}\times -\frac{\sqrt{2}}{2} < 0$

$f'(\frac{7\pi}{4}) = -2\sin(\frac{7\pi}{4})\cos(\frac{7\pi}{4}) = -2\times -\frac{\sqrt{2}}{2}\times \frac{\sqrt{2}}{2} > 0$

The function is decreasing before $x = \frac{3\pi}{2}$ and increasing after it, so $x = \frac{3\pi}{2}$ is a local minimum.

12. First find the critical numbers. Note the function's domain is all real numbers. Next, take a derivative.

$f'(x) = 5x^4 - 20$

The derivative's domain is also all real numbers. Set the derivative equal to zero and solve.

$0 = 5x^4 - 20 \to 0 = x^4 - 4 \to 0 = (x^2+2)(x^2-2) \to 0 = (x^2+2)(x+\sqrt{2})(x-\sqrt{2})$

Since $x^2 + 2 = 0$ has no solutions, the only critical numbers are $\pm\sqrt{2}$. To classify them, choose points below and above them, for instance, $x = -2, 0, 2$ and find the signs of the derivatives at these points.

$f'(-2) = 5(-2)^4 - 20 = 5\times 16 - 20 > 0$

$f'(0) = 5(0)^4 - 20 = -20 < 0$

$f'(2) = 5(2)^4 - 20 = 5\times 16 - 20 > 0$

Since the function is increasing before $x = -\sqrt{2}$ and decreasing after it, $x = -\sqrt{2}$ is a local maximum.

Since the function is decreasing before $x = \sqrt{2}$ and increasing after it, $x = \sqrt{2}$ is a local minimum.

13. Find the critical numbers by taking the derivative and setting it equal to zero.

$f'(x) = 1 + \cos(x) = 0 \to \cos(x) = -1$

On the interval $(-2\pi, 2\pi)$ the critical points are $x = \pm\pi$. Choose numbers to test between the critical points, for example, $x = -\frac{3\pi}{2}, 0, \frac{3\pi}{2}$. Find the sign of the derivative at these points.

$f'(-\frac{3\pi}{2}) = 1 + \cos(-\frac{3\pi}{2}) = 1 + 0 > 0$

$f'(0) = 1 + \cos(0) = 1 + 0 > 0$

$f'(\frac{3\pi}{2}) = 1 + \cos(-\frac{3\pi}{2}) = 1 + 0 > 0$

Since the function is increasing over every interval, the critical numbers do not represent maxima or minima. On the given interval the function has no maxima or minima.

14. Find critical points by taking the derivative and setting it equal to zero.

$$f'(x) = 3x^3 + 12x^2 - 12x - 48$$
$$0 = 3x^3 + 12x^2 - 12x - 48$$
$$0 = x^3 + 4x^2 - 4x - 16$$
$$0 = x^2(x+4) - 4(x+4) = (x^2 - 4)(x+4)$$
$$0 = (x+4)^2(x-4)$$

The critical numbers are $x = \pm 4$. Now choose values on every interval between and outside the critical values to test, for instance, $x = 0, x = \pm 5$. Test them by finding the sign of the second derivative.

$$f'(x) = 3(x+4)^2(x-4)$$
$$f'(-5) = 3(-5+4)^2(-5-4) < 0$$
$$f'(0) = 3(0+4)^2(0-4) < 0$$
$$f'(5) = 3(5+4)^2(5-4) > 0$$

Since the function is decreasing on either side of $x = -4$, that point is neither a maximum or minimum. Since the function decreases before and increases after $x = 4$, $x = 4$ is a local minimum. We know this function approaches $+\infty$ as $x$ approaches $\pm\infty$, so there is no global maximum, but there is a global minimum. To sum up, we find
local minimum: at $x = 4$
local maximum: none
global minimum: $x = 4$
global maximum: none

15. $f(x) = \dfrac{x^2 - x - 6}{x^2 + x - 6} = \dfrac{(x+2)(x-3)}{(x-2)(x+3)}$

The points $x = -3, 2$ are outside the domain. Now find the derivative.

$$f(x) = \frac{x^2 - x - 6}{x^2 + x - 6}$$
$$f'(x) = \frac{(2x-1)(x^2 + x - 6) - (2x+1)(x^2 - x - 6)}{(x^2 + x - 6)^2}$$
$$= \frac{(\cancel{2x^3} + 2x^2 \cancel{-12x} - x^2 \cancel{-x} + 6) - (\cancel{2x^3} - 2x^2 \cancel{-12x} + x^2 \cancel{-x} - 6)}{(x^2 + x - 6)^2}$$
$$= \frac{(2x^2 - x^2 + 6) - (-2x^2 + x^2 - 6)}{(x^2 + x - 6)^2} = \frac{2x^2 + 12}{(x^2 + x - 6)^2}$$

The $x$-values that make the denominator zero are excluded from the function's domain, so they are not critical numbers. No real $x$-values will make the numerator of the derivative (and therefore the derivative itself) zero, so the function has no critical numbers. Therefore it has no local extrema and no global extrema.

# The Second Derivative Test

**Answers**

1. There is a relative minimum at $x = 2;$ the relative minimum is located at (2,3)

2. $f(1) = 3$ suggests that $a + b = 2$ and $f'(1) = 0 = 2 + a;$ solving this system we have that $a = -2,\ b = 4$; the point (1,3) is an absolute max of $f$.

3. Relative maximum at $x = \frac{-2}{3}$, relative minimum at $x = 0$; the relative maximum is located at $(\frac{-2}{3}, 0.15)$; the relative minimum is located at $(0, 0)$. There is a point of inflection at $(-\frac{1}{3}, 0.07)$.

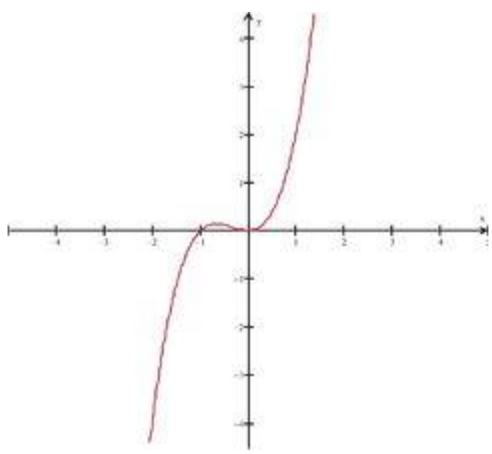

**Figure 4.5.4**

4. Relative maximum at $x = -\sqrt{3}$, located at $(-\sqrt{3}, -2\sqrt{3})$; relative minimum at $x = \sqrt{3}$, located at $(\sqrt{3}, 2\sqrt{3})$. There are no inflection points.

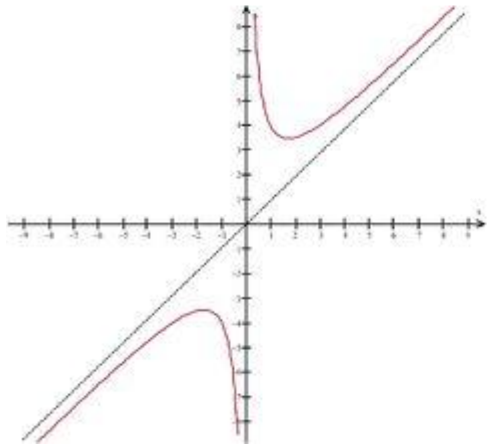

**Figure 4.5.5**

5. Relative maximum at $x = -2,$ relative minimum at $x = 2$; the relative maximum is located at $(-2, 16)$; the relative minimum is located at $(2, 16).$ There is a point of inflection at $(0, 0).$

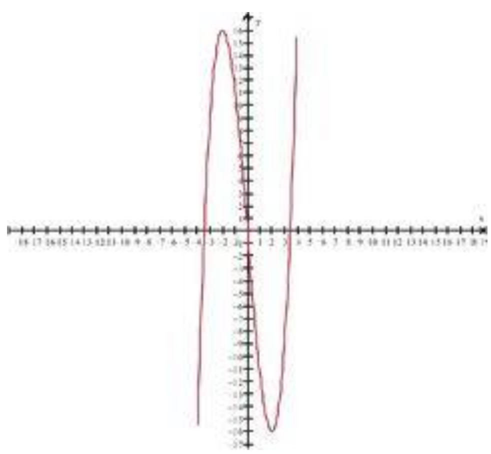

**Figure 4.5.6**

6. Relative maximums at $x = \pm 2,$ relative minimum at $x = 0$; the relative maximums are located at $(-2, 4)$ and $(2, 4)$; the relative minimum is located at $(0, 0).$

There are two inflection points, located at $(-\frac{2\sqrt{3}}{3}, \frac{20}{9})$ and $(\frac{2\sqrt{3}}{3}, \frac{20}{9}).$ The graph is concave up in the interval;

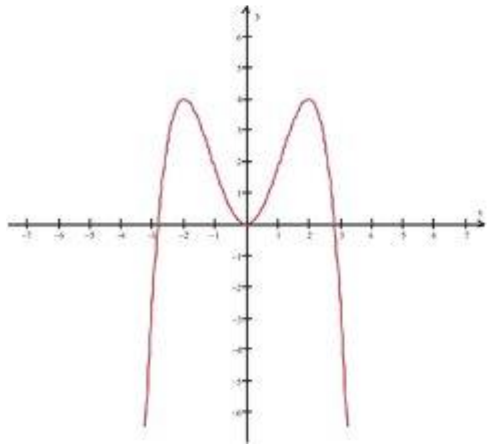

**Figure 4.5.7**

7. There is a relative minimum at $(0.25, -0.10)$.

8. False: there are inflection points at $x = 0$ and $x = -2$. There is a relative minimum at $x = -3$.

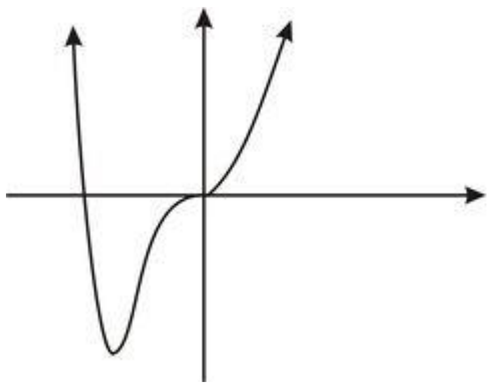

**Figure 4.5.8**

9. $f(x) = x^2 + (\frac{1}{x-1})$

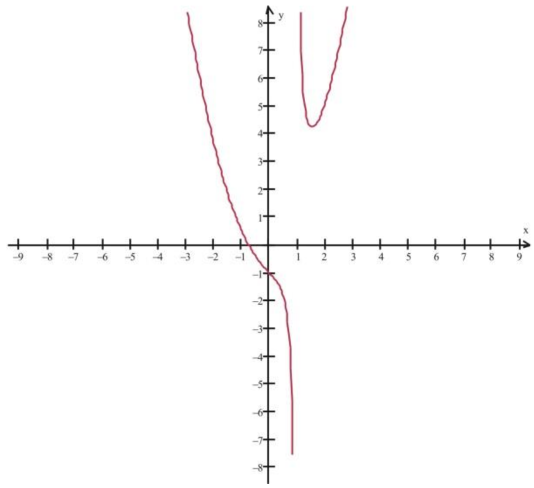

**Figure 4.5.9**

10. $f(x) = \sqrt{x}$ on $(0, +\infty)$

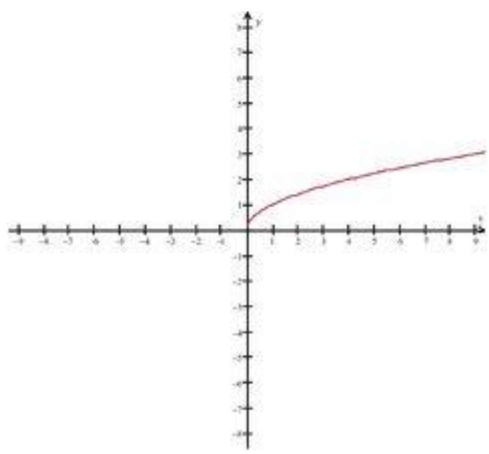

**Figure 4.5.10**

Also, $f(x) = -\dfrac{1}{x^2}$

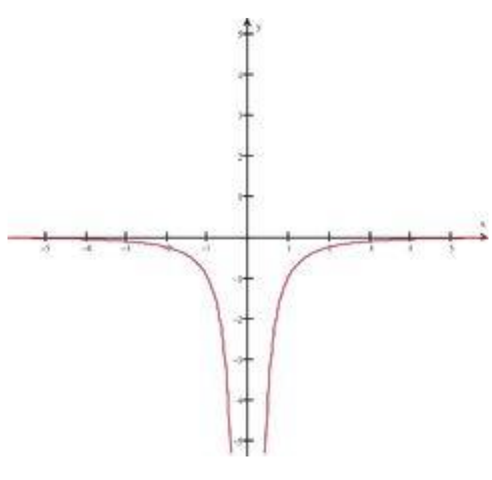

**Figure 4.5.11**

11. No, the first derivative does not provide any information about concavity. Only the second derivative may be used.

12. The second derivative is positive on the interval so the function is concave up.

$f(x) = x^4$

$f'(x) = 4x^3$

$f''(x) = 12x^2$

13. The second derivative is negative over the interval so the function is concave down.

$f(x) = -\cos(x)$

$f'(x) = \sin(x)$

$f''(x) = \cos(x)$

14. If the first derivative of $f(x)$ is $20^x \sin(\pi x)$, then the second derivative of $f(x)$ is $f''(x) = \pi 20^x \cos(\pi x) + \sin(\pi x) 20^x \ln(20)$. Cosine and sine are both positive on the interval $[0, \pi/2]$. The natural log of twenty is positive and $20^x$ is always positive, so the entire second derivative is positive. The function is concave up.

15. $x = -\frac{1}{4}$ is the only inflection point.

We need to take two derivatives.

$f(x) = 4x^3 + 3x^2 + 2x + 1$

$f'(x) = 12x^2 + 6x + 2$

$f''(x) = 24x + 6$

Now, we need to solve for when the second derivative is equal to zero.

$0 = 24x + 6 \to -6 = 24x \to x = \frac{-6}{24} = \frac{-1}{4}$

16. First we need to find the critical points of the function.

$f(x) = x^3 + x^2 - 5\ln(x) \to f'(x) = 3x^2 + 2x + \frac{-5}{x} = \frac{3x^3 + 2x^2 - 5}{x}$.

This is only zero when the polynomial in the numerator is equal to zero, which is when $x = 1$.

Now to apply the second derivative test we need to take yet another derivative.

$f'(x) = \frac{3x^3 + 2x^2 - 5}{x} \to f''(x) = \frac{9x^3 + 4x^2 - 3x^3 - 2x^2 + 5}{x^2} = \frac{6x^3 + 2x^2 + 5}{x^2}$

At $x = 1$ the second derivative is positive, so this is a minimum.

17. First we need to find the critical points of the function.

$f(x) = 5^{\sin(x)} \to f'(x) = \cos(x) 5^{\sin(x)} \ln(5)$.

The only zero is when cosine is equal to zero, which occurs at $x = \frac{\pi}{2}$.

Now we need to take a second derivative.

$f'(x) = \cos(x)5^{\sin(x)}\ln(5) \rightarrow f''(x) = -\ln(5)(\sin(x)5^{\sin(x)} + \cos^2(x)5^{\sin(x)}\ln(5))$

The interior of this last expression is always positive when sine is positive, which it is at $x = \frac{\pi}{2}$. Since we are then multiplying by the negative log of five, the second derivative is negative at $x = \frac{\pi}{2}$. Therefore we are dealing with an absolute minimum.

18. First we need to find the critical points of the function.

    $f(x) = \log(2x + x^2) \rightarrow f'(x) = \frac{2+2x}{2x+x^2}$

    The function is zero when the numerator is zero.

    $2 + 2x = 0 \rightarrow 2(1+x) = 0 \rightarrow x = -1$

    Now, we need to take the second derivative to classify the critical point.

    $f'(x) = \frac{2+2x}{2x+x^2} \rightarrow f''(x) = \frac{4x+2x^2-4-8x-4x^2}{(2x+x^2)^2} = \frac{-2x^2-4x-4}{(2x+x^2)^2} = -2\frac{x^2+2x+2}{(2x+x^2)^2}$

    At negative one, this function is negative, so the point is a local maximum.

19. The first derivative of this function is $f'(x) = 4x^3$. There are no critical points and so no extrema.

20. No, we cannot.

    $f(x) = x^{10} + 25$

    $f'(x) = 10x^9$

    $f''(x) = 90x^8$

    At the given point both the first and second derivatives are zero, so we do not have any information.

# Using the First and Second Derivative Tests

## Answers

1. $f'(x) = 3x^2 - 6x = 0$ means critical points at $x = 0, 2$

   $f''(x) = 6x - 6$: $f''(0) = -6$ means concave down and local maximum.

   $f''(2) = 6$ means concave up and local minimum.

   $f''(x) = 0$ at $x = 1$, a point of inflection.

2. $f'(x) = 4x^3 - 6x^2 - 1 = 0$ means critical points at $x = 1.60$

   $f''(x) = 12x^2 - 12x$: $f''(1.60) > 0$ means concave up and local minimum.

   $f''(x) = 0$ at $x = 0,1$ are points of inflection

3. $f'(x) = 2\cos 2x = 0$ at $x = n\dfrac{\pi}{4}, n = 1,3$ .

   $f''(x) = -4\sin 2x$: $f''(\pi/2) = -4$ means concave down and a local maximum

   $f''(x = 3\pi/4) = 4$ means concave up and a local minimum

   $f''(x) = -4\sin 2x = 0$ at $x = 0, \pi/2, \pi$ are inflection points.

4. $f'(x) = \dfrac{-(2x-1)}{(x^2 - x + 2)^2} = 0$ means critical points at $x = \dfrac{1}{2}$

   $f''(x) = \dfrac{2(11x^2 - 11x - 1)}{(x^2 - x + 2)^3}$: $f''(0.5) < 0$ means concave down and a local maximum.

   $f''(x) = \dfrac{2(11x^2 - 11x - 1)}{(x^2 - x + 2)^3} = 0$ at $x = \dfrac{11 \pm \sqrt{165}}{22}$ which are inflection points.

5. $f'(x) = 6(x^2 - x + 1) = 0$ means non-real critical points of $x$, so there will not be real local minima or maxima .

   $f'(x) = 6(2x - 1) = 0$ at $x = \dfrac{1}{2}$ which is an inflection point.

6. $f(x) = x^3 - 12x + 5$ in the interval [-5,3] $f'(x) = 3x^2 - 12 = 0$ at $x = \pm 2$ .

   $f''(x) = 6x$: $f''(-2) < 0$ which means concave down and a local maximum.

   $f''(2) > 0$ which means concave up and a local minimum.

   $f''(x) = 6x = 0$ at $x = 0$ which means this is an inflection point.

7. $f(x)=x^5+20x+5 \quad f'(x)=5x^4+20=0$ at $x=\pm\sqrt{2}$

$f''(x)=20x^3$: $f''(-\sqrt{2})<0$ which means concave down and a local maximum.

$f''(\sqrt{2})>0$ which means concave up and a local minimum.

$f''(x)=20x^3=0$ at $x=0$ which is an inflection point.

8. $f(x) = x^2 - 5x + 6, \ (-1, 3] \quad f'(x) = 2x - 5 = 0$ at $x=\frac{5}{2}$ .

$f''(x) = 2$: which means concave up and a local minimum at $x=\frac{5}{2}$.

There are no inflection points

9. $f'(x)=3x^2-6x=0$ at $x=0,2$, and need to check end points of [0,4].

$f''(x)=6x-6$: $f''(0)<0$ means concave down and a local maximum.

$f''(2)>0$ means concave up and a local minimum.

$f(0)=2$, $f(2)=-2$ and $f(4)=18$

Absolute maximum at $f(4)=18$ and absolute minimum at $f(2)=-2$.

10. $f'(x)=xe^{-x}(-x+2)$ with critical points at $x=0,2$

$f''(x)=e^{-x}(x^2-4x+2)$: $f''(0)>0$ means concave up and a local minimum.

$f''(2)<0$ means concave down and a local maximum

. $f''(x)=e^{-x}(x^2-4x+2)=0$ at $x=2\pm\sqrt{2}$ are inflection points.

11. $f'(x)=\frac{x(x-2)}{(x-1)^2}=0$ with critical points at $x=0,1,2$ with $x=1$ not in the domain.

$f''(x)=\frac{2}{(x-1)^3}$: $f''(0)<0$ means concave down and a local maximum

$f''(2)>0$ means concave up and a local minimum.

There are no inflection points.

12. Use a calculator to find $f'(x)=-5e^{-x}+3x^2=0$ at critical point $x=0.8458$ .

This is a local (and global) minimum. There is no inflection point.

13. There is a critical point at $x = \frac{\ln 2}{3}$ which is a global minimum ( $f'' > 0$ ); there is no inflection point.

14. No critical points since $f'(x) = 3^x e^{-x}(\ln 3 - 1)$ is defined everywhere but has no 0.

15. On the interval $[-\pi, \pi]$ there are critical points at $x = 0$, where $f'(x) = x\sec^2 x + \tan x = 0$, and $x = \pm\pi/2$ where $f'(x)$ is not defined;

$f''(0) = 2$, so this is a local minimum; and $f''(\pm\pi/2)$ is not defined.

# Evaluating Indeterminate Limits: L'Hospital's Rule

Answers

1. $\lim_{x\to 3}\dfrac{x^2-9}{x-3}=6$

2. $\lim_{x\to 0}\dfrac{\sqrt{1+x}-\sqrt{1-x}}{x}=1$

3. $\lim_{x\to +\infty}\dfrac{\ln(x)}{\sqrt{x}}=0$

4. $\lim_{x\to +\infty} x^2e^{-2x}=0$

5. $\lim_{x\to 0}(1-x)^{\frac{1}{x}}=\dfrac{1}{e}$ ; Hint: Let $(1-x)^{\frac{1}{x}}=e^{\ln(1-x)^{\frac{1}{x}}}$, so $\lim_{x\to 0}(1-x)^{\frac{1}{x}}=\lim_{x\to 0}e^{\ln(1-x)\frac{1}{x}}=e^{\lim_{x\to 0}\frac{1}{x}\ln(1-x)}$

6. $\lim_{x\to 0}\dfrac{e^x-1-x}{x^2}=\dfrac{1}{2}$

7. $\lim_{x\to -\infty}\dfrac{e^x-1-x}{x^2}=0$

8. $\lim_{x\to \infty} x^{\frac{1}{4}}\ln(x)=0$

9. $\lim_{x\to \pi/2}\dfrac{\tan x}{1+\tan x}=1$

10. $\lim_{x\to 0}\dfrac{\tan x}{x}=1$

11. $\lim_{x\to 0}\dfrac{x^2}{\sin^2 x}=1$

12. $\lim_{x\to 0}\dfrac{e^{x^2}-1}{\sin x^2}=1$

13. $\lim_{x\to\infty} \frac{\sqrt{x^2+1}}{x} = 1$

14. $\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$

15. $\lim_{x\to\infty} x^3 e^{-x} = \lim_{x\to\infty} \frac{x^3}{e^x} = 0$, Hint: Apply L'Hospital's Rule 3 times.

# Analyzing the Graphs of Functions

## Answers

***Table Summary***

| **1.** $f(x) = x^3 + 3x^2 - x - 3$ | ***Analysis*** |
|---|---|
| ***Domain and Range*** | $D = (-\infty, +\infty)$,<br>$R = \{\text{all reals}\}$ |
| ***Intercepts and Zeros*** | Zeros at $x = \pm 1, -3,$<br>$y-$intercept at $(0, -3)$ |
| ***Asymptotes and limits at infinity*** | No asymptotes |
| ***Differentiability*** | Differentiable at every point of its domain |
| ***Intervals where*** $f$ ***is increasing*** | $\left(-\infty, \frac{-3-2\sqrt{3}}{3}\right)$ and $\left(\frac{-3+2\sqrt{3}}{3}, +\infty\right)$ |
| ***Intervals where*** $f$ ***is decreasing*** | $\left(\frac{-3-2\sqrt{3}}{3}, \frac{-3+2\sqrt{3}}{3}\right)$ |
| ***Relative extrema*** | Relative maximum at $x = \frac{-3-2\sqrt{3}}{3}$, located at the point $(-2.15, 3.07)$;<br>Relative minimum at $x = \frac{-3+2\sqrt{3}}{3}$, located at the point $(0.15, -3.07)$ |

***Table Summary***

| **2.** $f(x) = -x^4 + 4x^3 - 4x^2$ | ***Analysis*** |
|---|---|
| ***Domain and Range*** | $D = (-\infty, +\infty)$,<br>$R = \{y \leq 0\}$ |
| ***Intercepts and Zeros*** | Zeros at $x = 0, 2,$<br>$y-$intercept at $(0, 0)$ |
| ***Asymptotes and limits at infinity*** | No asymptotes |
| ***Differentiability*** | Differentiable at every point of its domain |
| ***Intervals where*** $f$ ***is increasing*** | $(-\infty, 0)$and $(1, 2)$ |
| ***Intervals where*** $f$ ***is decreasing*** | $(0, 1)$and $(2, +\infty)$ |
| ***Relative extrema*** | Relative maximum at $x = 0$, located at the point $(0, 0)$; at $x = 2$, located at the point $(2, 0)$<br>Relative minimum at $x = 1$, located at the point $(1, -1)$ |
| ***Concavity*** | Concave up in $\left(1 - \frac{\sqrt{3}}{3}, 1 + \frac{\sqrt{3}}{3}\right)$<br>Concave down in $\left(-\infty, 1 - \frac{\sqrt{3}}{3}\right)$ and $\left(1 + \frac{\sqrt{3}}{3}, +\infty\right)$ |
| ***Inflection points*** | $x = 1 + \frac{\sqrt{3}}{3}, 1 - \frac{\sqrt{3}}{3}$, located at the points $(1.577, -0.444)$and $(0.423, -0.444)$ |

***Table Summary***

| **3.** $f(x) = \frac{2x-2}{x^2}$ | ***Analysis*** |
|---|---|
| ***Domain and Range*** | $D = (-\infty, 0) \cup (0, +\infty)$,<br>$R = \{y \neq 0\}$ |
| ***Intercepts and Zeros*** | Zeros at $x = 1$,<br>No $y-$intercept |
| ***Asymptotes and limits at infinity*** | HA $y = 0$ |
| ***Differentiability*** | Differentiable at every point of its domain |
| ***Intervals where $f$ is increasing*** | $(0, 2)$ |
| ***Intervals where $f$ is decreasing*** | $(-\infty, 0)$and $(2, +\infty)$ |
| ***Relative extrema*** | Relative maximum at $x = 2$,located at the point $(2, 0.5)$ |
| ***Concavity*** | Concave up in $(3, +\infty)$<br>Concave down in $(-\infty, 0)$and $(0, 3)$ |
| ***Inflection points*** | $x = 3$,located at the point $(3, \frac{4}{9})$ |

***Table Summary***

| **4.** $f(x) = x - x^{\frac{1}{3}}$ | ***Analysis*** |
|---|---|
| ***Domain and Range*** | $D = (-\infty, +\infty)$,<br>$R = \{\text{all reals}\}$ |
| ***Intercepts and Zeros*** | Zeros at $x = \pm 1, 0,$ |

***Table Summary***

| **4.** $f(x) = x - x^{\frac{1}{3}}$ | ***Analysis*** |
|---|---|
| | $y-$intercept at $(0,0)$ |
| ***Asymptotes and limits at infinity*** | No asymptotes |
| ***Differentiability*** | Differentiable in $(\infty,0)\cup(0,+\infty)$ |
| ***Intervals where*** $f$ ***is increasing*** | $\left(-\infty,\frac{-\sqrt{3}}{9}\right)$and $\left(\frac{\sqrt{3}}{9},+\infty\right)$ |
| ***Intervals where*** $f$ ***is decreasing*** | $\left(\frac{-\sqrt{3}}{9},\frac{\sqrt{3}}{9}\right)$ |
| ***Relative extrema*** | Relative maximum at $x=\frac{-\sqrt{3}}{9}$,located at the point $\left(\frac{-\sqrt{3}}{9},0.384\right)$<br><br>Relative minimum at $x=\frac{\sqrt{3}}{9}$,located at the point $\left(\frac{\sqrt{3}}{9},-0.384\right)$ |
| ***Concavity*** | Concave up in $(0,+\infty)$<br><br>Concave down in $(-\infty,0)$ |
| ***Inflection points*** | $x=0$, located at the point $(0,0)$ |

***Table Summary***

| **5.** $f(x) = -\sqrt{2x-6}+3$ | ***Analysis*** |
|---|---|
| ***Domain and Range*** | $D=(3,+\infty)$,$R=\{y\leq 3\}$ |

***Table Summary***

| **5.** $f(x) = -\sqrt{2x-6}+3$ | ***Analysis*** |
|---|---|
| ***Intercepts and Zeros*** | Zero at $x = \frac{15}{2}$,<br>No $y-$intercept |
| ***Asymptotes and limits at infinity*** | No asymptotes |
| ***Differentiability*** | Differentiable in $(3, +\infty)$ |
| ***Intervals where*** $f$ ***is increasing*** | Nowhere |
| ***Intervals where*** $f$ ***is decreasing*** | Everywhere in $D = (3, +\infty)$ |
| ***Relative extrema*** | None<br>Absolute maximum at $x = 3$, located at $(3, 3)$ |
| ***Concavity*** | Concave up in $(3, +\infty)$ |
| ***Inflection points*** | None |

***Table Summary***

| **6.** $f(x) = x^2 - 2\sqrt{x}$ | ***Analysis*** |
|---|---|
| ***Domain and Range*** | $D = [0, +\infty)$,<br>$R = \{\text{all reals } \geq -1.19\}$ |
| ***Intercepts and Zeros*** | Zeros at $x = 0$ and $x = \sqrt[3]{4}$,<br>$y-$intercept at $(0, 0)$ |
| ***Asymptotes and limits at infinity*** | No asymptotes |
| ***Differentiability*** | Differentiable in $(0, +\infty)$ |
| ***Intervals where*** $f$ ***is increasing*** | $\left(\frac{\sqrt[3]{16}}{4}, +\infty\right)$ |
| ***Intervals where*** $f$ ***is decreasing*** | $\left(0, \frac{\sqrt[3]{16}}{4}\right)$ |
| ***Relative extrema*** | Relative minimum at $x = \frac{\sqrt[3]{16}}{4}$, located at the point $x = \left(\frac{\sqrt[3]{16}}{4}, -1.19\right)$ |
| ***Concavity*** | Concave up in $(0, +\infty)$ |
| ***Inflection points*** | None |

*Table Summary*

| **7.** $f(x) = 1 + \cos x$ | ***Analysis*** |
|---|---|
| ***Domain and Range*** | $D = [-\pi, \pi]$,<br>$R = \{0 \leq y \leq 2\}$ |
| ***Intercepts and Zeros*** | Zeros at $x = -\pi, \pi$,<br>$y-$intercept at $(0, 2)$ |
| ***Asymptotes and limits at infinity*** | No asymptotes; $\lim_{x \leftrightarrow \infty} f(x)$ does not exist |
| ***Differentiability*** | Differentiable at every point of its domain |
| ***Intervals where $f$ is increasing*** | $(-\pi, 0)$ |
| ***Intervals where $f$ is decreasing*** | $(0, \pi)$ |
| ***Relative extrema*** | Absolute maximum at $x = 0$, located at the point $(0, 2)$<br>Absolute minimums at $x = \pm\pi$, located at the points $(-\pi, 0)$ and $(\pi, 0)$ |
| ***Concavity*** | Concave down in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$,<br>Concave up in $\left(-\pi, -\frac{\pi}{2}\right)$ and $\left(\frac{\pi}{2}, \pi\right)$ |
| ***Inflection points*** | $x = \pm\frac{\pi}{2}$, located at the points $\left(-\frac{\pi}{2}, 1\right)$ and $\left(\frac{\pi}{2}, 1\right)$ |

***Table Summary***

| **8.** $f(x)=x^2-x+1$ | ***Analysis*** |
|---|---|
| ***Domain and Range*** | $D=(-\infty,+\infty)$<br>$R=(\frac{3}{4},+\infty)$ |
| ***Intercepts and Zeros*** | Non-real zeros<br>y-intercept (0,1) |
| ***Asymptotes and limits at infinity*** | No asymptotes<br>$\lim_{x\to\pm\infty} f(x)=\infty$ |
| ***Differentiability*** | Differentiable at every point |
| ***Intervals where*** $f$ ***is increasing*** | $(0.5,\infty)$ |
| ***Intervals where*** $f$ ***is decreasing*** | $(-\infty,0.5)$ |
| ***Relative extrema*** | Global minimum at (0.5, 0.75) |
| ***Concavity*** | Concave up everywhere |
| ***Inflection points*** | None |

***Table Summary***

| **9.** $f(x)=4x^3-6x^2-1$ | ***Analysis*** |
|---|---|
| ***Domain and Range*** | $D=(-\infty,+\infty)$<br><br>$R=(-\infty,+\infty)$ |
| ***Intercepts and Zeros*** | Zero at $x=1.5979$<br><br>y-intercept (0,-1) |
| ***Asymptotes and limits at infinity*** | No asymptotes<br><br>$\lim_{x\to+\infty} f(x)=\infty$<br><br>$\lim_{x\to-\infty} f(x)=-\infty$ |
| ***Differentiability*** | Differentiable at every point |
| ***Intervals where $f$ is increasing*** | $(-\infty,0)\cup(1,\infty)$ |
| ***Intervals where $f$ is decreasing*** | $(0,1)$ |
| ***Relative extrema*** | Local maximum at (0, 1) |
| ***Concavity*** | Concave up in $(-\infty,0)$ and $(1,\infty)$.<br><br>Concave down in $(0,1)$. |
| ***Inflection points*** | At the point $(0.5,-2)$ |

***Table Summary***

| **10.** $f(x)=\frac{x^2}{x-1}$ | ***Analysis*** |
|---|---|
| ***Domain and Range*** | $D=(x\neq 1)$<br><br>$R=(-\infty,1)$ and $(1,\infty)$ |
| ***Intercepts and Zeros*** | Zero at $x=0$<br><br>y-intercept (0,0) |
| ***Asymptotes and limits at infinity*** | VA at $x=1$<br><br>$\lim_{x\to+\infty} f(x)=\infty$<br><br>$\lim_{x\to-\infty} f(x)=-\infty$ |
| ***Differentiability*** | Differentiable at all points except $x=1$ |
| ***Intervals where $f$ is increasing*** | $(-\infty,0)\cup(2,\infty)$ |
| ***Intervals where $f$ is decreasing*** | $(0,1)$ and $(1,2)$ |
| ***Relative extrema*** | Local maximum at (0, 0) in $(-\infty,1)$.<br><br>Local minimum at (2,4) in $(1,\infty)$ |
| ***Concavity*** | Concave down in $(-\infty,1)$.<br><br>Concave up in $(1,\infty)$. |
| ***Inflection points*** | None |

***Table Summary***

| **11.** $f(x)=x^2e^x$ | ***Analysis*** |
|---|---|
| ***Domain and Range*** | $D=(-\infty,+\infty)$<br><br>$R=(0,\infty)$ |
| ***Intercepts and Zeros*** | Zero at $x=0$<br><br>y-intercept (0,0) |
| ***Asymptotes and limits at infinity*** | HA at $y=0$<br><br>$\lim_{x\to+\infty} f(x)=0$<br><br>$\lim_{x\to-\infty} f(x)=\infty$ |
| ***Differentiability*** | Differentiable at every point |
| ***Intervals where $f$ is increasing*** | $(0,2)$ |
| ***Intervals where $f$ is decreasing*** | $(-\infty,0)$ and $(2,\infty)$ |
| ***Relative extrema*** | Local maximum at $(2,4e^{-2}=0.5413)$<br><br>Minimum at (0,0) |
| ***Concavity*** | Concave up in $(-\infty,2-\sqrt{2})$ and $(2+\sqrt{2},\infty)$<br><br>Concave down in $(2-\sqrt{2},2+\sqrt{2})$. |
| ***Inflection points*** | $x=2\pm\sqrt{2}$ |

***Table Summary***

| **12.** $f(x) = \cos x - x$ | ***Analysis*** |
|---|---|
| ***Domain and Range*** | $D = (-\infty, +\infty)$<br>$R = (-\infty, \infty)$ |
| ***Intercepts and Zeros*** | Zero at $x = 0.7391$<br>y-intercept (0,1) |
| ***Asymptotes and limits at infinity*** | No asymptotes<br>$\lim_{x \to +\infty} f(x) = -\infty$<br>$\lim_{x \to -\infty} f(x) = \infty$ |
| ***Differentiability*** | Differentiable at every point |
| ***Intervals where $f$ is increasing*** | None |
| ***Intervals where $f$ is decreasing*** | $(-\infty, \infty)$ |
| ***Relative extrema*** | None |
| ***Concavity*** | Concave up in $(\frac{\pi}{2} + 2\pi n, \frac{3\pi}{2} + 2\pi n)$<br>Concave down in $(-\frac{\pi}{2} + 2\pi n, \frac{\pi}{2} + 2\pi n)$ . |
| ***Inflection points*** | $x = n\frac{\pi}{2}$ where $n = \pm 1, \pm 3, \pm 5, \ldots$ |

***Table Summary***

| **13.** $f(x)=e^{-2x}+e^{x}$ | ***Analysis*** |
|---|---|
| ***Domain and Range*** | $D=(-\infty,+\infty)$<br>$R=(1.89,\infty)$ |
| ***Intercepts and Zeros*** | No zeros<br>y-intercept (0,2) |
| ***Asymptotes and limits at infinity*** | No asymptotes<br>$\lim_{x\to+\infty} f(x)=\infty$<br>$\lim_{x\to-\infty} f(x)=\infty$ |
| ***Differentiability*** | Differentiable at every point |
| ***Intervals where $f$ is increasing*** | $(\frac{\ln 2}{3},\infty)$ |
| ***Intervals where $f$ is decreasing*** | $(-\infty,\frac{\ln 2}{3})$ |
| ***Relative extrema*** | Minimum at $(\frac{\ln 2}{3},1.89)$ |
| ***Concavity*** | Concave up everywhere |
| ***Inflection points*** | No inflection points |

***Table Summary***

| **14.** $f(x)=5e^{-x}+x^3$ | ***Analysis*** |
|---|---|
| ***Domain and Range*** | $D=(-\infty,+\infty)$<br>$R=(2.75,\infty)$ |
| ***Intercepts and Zeros*** | No zeros<br>y-intercept (0,5) |
| ***Asymptotes and limits at infinity*** | No asymptotes<br>$\lim_{x\to+\infty} f(x)=\infty$<br>$\lim_{x\to-\infty} f(x)=\infty$ |
| ***Differentiability*** | Differentiable at every point |
| ***Intervals where*** $f$ ***is increasing*** | $(0.8458,\infty)$ |
| ***Intervals where*** $f$ ***is decreasing*** | $(-\infty,0.8458)$ |
| ***Relative extrema*** | Minimum at $(0.8458,2.75)$ |
| ***Concavity*** | Concave up everywhere |
| ***Inflection points*** | No inflection points |

***Table Summary***

| **15.** $f(x)=x^5-7x^2+2$ | ***Analysis*** |
|---|---|
| ***Domain and Range*** | $D=(-\infty,+\infty)$<br>$R=(-\infty,+\infty)$ |
| ***Intercepts and Zeros*** | Zeros at $x=-0.529,\ 0.541,\ 1.859$<br>y-intercept (0,2) |
| ***Asymptotes and limits at infinity*** | No asymptotes<br>$\lim_{x\to+\infty} f(x)=\infty$<br>$\lim_{x\to-\infty} f(x)=-\infty$ |
| ***Differentiability*** | Differentiable at every point |
| ***Intervals where $f$ is increasing*** | $(-\infty,0)\cup(1.41,\infty)$ |
| ***Intervals where $f$ is decreasing*** | $(0,1.41)$ |
| ***Relative extrema*** | Local maximum at (0, 2)<br>Local minimum at (1.41, -6.344) |
| ***Concavity*** | Concave down in $(-\infty,0.888)$<br>Concave up in $(0.888,\infty)$ . |
| ***Inflection points*** | At the point $(0.888,-2.97)$ |

# Optimization

Answers

1. Absolute minimum at $x = \frac{3}{2}, f\left(\frac{3}{2}\right) = \frac{3}{2}$. Absolute maximum at $x = 5, \quad f(5) = 26.$
2. Absolute minimum at $x = 0, \quad f(0) = 0.$ Absolute maximum at $x = 3, \quad f(3) = 54.$
3. Absolute minimum at $x = 8, \quad f(8) = -30.$ Absolute maximum at $x = 1, \quad f(1) = 3.$
4. Absolute minimum at $x = 0.75, \quad f(0.75) = -0.105.$ Absolute maximum at $x = -2,$ $f(-2) = 24.$
5. $x = y = 20\sqrt{5}$
6. $x = y = 5\sqrt{2}$
7. At $t = 20$ ft, the basketball will reach a height of $s(t) = 25$ ft.
8. The rocket will take approximately $t \approx 10.4$ sec to attain its maximum height of $321.7$ ft. the rocket will hit the ground at $t \approx 16.6$ sec.
9. Assuming that the given perimeter is a constant $P$:
   $P = 2w + 2l$, where $w$ is the width of the rectangle and $l$is the length.

   The area of this rectangle is given by $A = Wl$. Since we want to maximize the area, we first isolate $w$ in the perimeter equation: $w = \frac{P-2l}{2}$
   And then substitute this into the area equation: $A = lw = l\left(\frac{P-2l}{2}\right) = \frac{Pl}{2} - l^2$.

   To find the maximum of $A$, we then take the derivative of $A$ with respect to $l$, set the derivative equal to zero, and solve for $l$ and the associated critical value:
   $\frac{dA}{dl} = \frac{P}{2} - 2l = 0$
   Which means that: $2l = \frac{P}{2}, or\ l = \frac{P}{4},$ will produce either a maximum or minimum for the area $A$.

   We also take the second derivative of $A$with respect to $l$ to determine whether this $l = \frac{P}{4}$ produces a maximum or a minimum: $\frac{d^2A}{dl^2} = -2$. The negative value for the second derivative over the entire domain of $A$ means that the function A is concave downwards everywhere, which, in turn, means that the function $A$ at the point $l = \frac{P}{4}$ is a maximum.

   Finally, since $l = \frac{P}{4}$, we know that: $w = \frac{P-2l}{2} = \frac{P-2\left(\frac{P}{4}\right)}{2} = \frac{\frac{P}{2}}{2} = \frac{P}{4}$.

   Since $w = l = \frac{P}{4}$, we have shown that the dimensions that produce the maximum area are square dimensions.

10. Assume that the given area is a constant $A$. Then: $A = wl$, where $w$is the width of the rectangle and $l$is the length. Then the perimeter of this rectangle is given by $P = 2w + 2l$.
Since we want to minimize the perimeter, we first isolate $w$in the area equation: $w = \frac{A}{l}$.
Substituting this into the perimeter equation we get: $P = 2w + 2l = 2\left(\frac{A}{l}\right) + 2l$. To find the point of minimization of this perimeter function, we take the derivative of this function with respect to $l$ and set it to zero and solve for $l$:

$$\frac{dP}{dl} = \frac{-2A}{l^2} + 2 = 0$$
$$2 = \frac{2A}{l^2}$$
$$l^2 = A$$
$$l = \sqrt{A}$$

To find $w$we substitute this value back into the area formula:

$$A = wl = w\sqrt{A}$$
$$W = \frac{A}{\sqrt{A}} = \sqrt{A}$$

So we have either a perimeter maximum or a minimum at $w = l = \sqrt{A}$. To determine which, we look at the second derivative of the perimeter function: $\frac{d^2P}{dl^2} = \frac{4A}{l^3}$. Since we know that the domain of $l$ is limited to positive values, then $\frac{d^2P}{dl^2} = \frac{4A}{l^3}$ is positive for all values of $l$, and the function $P$ is concave upwards everywhere in the domain for $l$, and so $w = l = \sqrt{A}$produces an absolute minimum for $P$.

Accordingly, we have shown that the smallest perimeter is in the shape of a square.

11. Twenty five feet by twenty five feet.
The formula for the area of a rectangle is $A = lw$. Since there is only 100 feet of fence, we have that $2l + 2w = 100 \rightarrow l + w = 50 \rightarrow l = 50 - w$. Area can therefore be reformulated as $A = (50 - w)w = 50w - w^2$.
To find the maximum area we need to take the derivative of the area function and set it equal to zero, which would give us $50 - 2w = 0 \rightarrow 50 = 2w \rightarrow w = 25$.
Therefore, the length and the width of the fence should both be twenty five—a square!

12. Eighteen and eighteen.
Let the two numbers Hans is after be $x$ and $y$. Then $x + y = 36 \rightarrow x = 36 - y$. We are trying to maximize $xy = y(36 - y) = 36y - y^2$. Taking the derivative and setting it to zero gives $36 - 2y = 0 \rightarrow 36 = 2y \rightarrow y = 18$. The two numbers, then, are eighteen and eighteen.

13. Let the two numbers she is after be $x$ and $y$. Then $xy = 25 \rightarrow y = \frac{25}{x}$. Without loss of generality, we are trying to maximize $x + y^2 = x + \frac{25^2}{x^2}$. Taking the derivative of this function and setting it to zero gives $1 - 2\frac{25^2}{x^3} = 0 \rightarrow \frac{1}{2} = \frac{25^2}{x^3} \rightarrow x^3 = 1250 \rightarrow x = 10.77$, which means that $y = 2.32$.

    The function we are trying to maximize is equal to 16.15. Since five and five give a value of thirty, this is a minimum.

14. Because she only has fifty feet of fence and one side is enclosed for free, we have that $50 = 2x + y$. The function we are trying to maximize is $A = xy$. Since $y = 50 - 2x$, we can rewrite this as $x(50 - 2x) = 50x - 2x^2$. Taking the derivative and setting it to zero yields $50 - 4x = 0 \rightarrow 50 = 4x \rightarrow 25 = 2x \rightarrow x = \frac{25}{2}$. Therefore $y = 25.$, and we are done.

15. The distance between two points is equal to $\sqrt{(x_0 - x_1)^2 + (y_0 - y_1)^2}$, but instead of minimizing the distance we can minimize the square of the distance, as this will not change the answer. Therefore, we need two numbers so that $y = \sqrt{x}$ and that $(1-x)^2 + y^2$ is minimized. Substituting in the first equation into the second gives $(1 - y^2)^2 + y^2$, taking the derivative and setting it equal to zero gives:

    $$\frac{d}{dx}((1-y^2)^2 + y^2 = \frac{d}{dx}(1 - y^2 + y^4) = -2y + 4y^3 = 2y(2y^2 - 1) = 0$$

    The positive roots of this equation are $y = 0, y = \frac{1}{\sqrt{2}}$ Plugging both into the distance formula and checking which distance is smaller gives as the minimum $(\frac{1}{\sqrt[4]{2}}, \frac{1}{\sqrt{2}})$.N

# Linearization of a Function

**Answers**

1. $f(x) \approx 2$
2. $f(x) \approx x - \pi$
3. Hint: Let $x = 0.$
4. 
   a. $1 - 4x$
   b. $1 - \frac{1}{2}x$
   c. $5 + 5x$
   d. $-1 + 2x$
   e. $1.297$

5. $f(0.3) \approx 1.3$ with linearization $f(x) \approx x+1$; calculator value 1.2955
6. $f(0.006) \approx 1.003$ with linearization $f(x) \approx \frac{1}{2}x+1$; calculator value 1.002996
7. $f(0.02) \approx 1.2$ with linearization $f(x) \approx 10x+1$; calculator value 1.219
8. $f(-0.115) \approx 1.0575$ with linearization $f(x) \approx -\frac{1}{2}x+1$; calculator value 1.0630
9. $f(3.65) \approx 11.95$ with linearization $f(x) \approx 3x+1$; calculator value 12.3725
10. $f(3.3) \approx 37.1$ with linearization $f(x) \approx 27x-52$; calculator value 37.937
11. $f(2.03) \approx 0.2425$ with linearization $f(x) \approx -\frac{1}{4}x+\frac{3}{4}$; calculator value 0.24267
12. $f(0.2) \approx 1.2$ with linearization $f(x) \approx x+1$; calculator value 1.2214
13. Linearize $f(x) = \sqrt{x}$ around 625: $f(628) \approx 5.006$ with linearization

    $f(x) \approx \frac{x}{500} + 3.75$; calculator value 5.00599

14. Linearize $f(x) = \frac{1}{2+x}$ ; $f(0.007) \approx 0.49825$ with linearization

    $f(x) \approx -\frac{x}{4} + 0.5$; calculator value 0.498256.

15. $f(1.3) \approx 2.42426$; the linearization around $x=1$ is $f(x) \approx \sqrt{2}x + (2-\sqrt{2})$.

# Approximating Roots of a Function: Newton's Method

## Answers

1. $x \approx -1.442$

2. $x \approx 1.146$ and $x \approx 7.854$

3. For this problem $x_{n+1} = x_n - \dfrac{4x_n^{\ 2} - x_n - 2}{8x_n - 1}$. With $x_0 = 0$, $x_5 = -59309$

4. For this problem $x_{n+1} = x_n - \dfrac{4x_n^{\ 3} - 6x_n^{\ 2} - 1}{12x_n^{\ 2} - 12x_n}$. With $x_0 = 1.5$, $x_3 = 1.5979$

5. For this problem $x_{n+1} = x_n - \dfrac{-5e^{-x_n} + 3x_n^{\ 2}}{5e^{-x_n} + 6x_n}$. With $x_0 = 0$, $x_3 = 0.8458$

6. For this problem $x_{n+1} = x_n - \dfrac{\cos(x_n) - x_n}{-\sin(x_n) - 1}$. With $x_0 = 0$, $x_3 = 0.7391$

7. For this problem $x_{n+1} = x_n - \dfrac{x_n^{\ 5} - 7x_n^{\ 2} + 2}{5x_n^{\ 4} - 14x_n}$. With $x_0 = -1$, $x_4 = -0.52896$

8. For this problem $x_{n+1} = x_n - \dfrac{x_n^{\ 5} - 7x_n^{\ 2} + 2}{5x_n^{\ 4} - 14x_n}$. With $x_0 = 1$, $x_3 = 0.54066$

9. For this problem $x_{n+1} = x_n - \dfrac{x_n^{\ 2}\cos x_n - x_n}{-x_n(x_n \sin x_n - 2\cos x_n) - 1_n}$. With $x_0 = 5$, $x_2 = 4.91719$

10. For this problem $x_{n+1} = x_n - \dfrac{x_n^{\ 2}\cos x_n - x_n}{-x_n(x_n \sin x_n - 2\cos x_n) - 1_n}$. With $x_0 = 8$, $x_3 = 7.72415$

11. For this problem $x_{n+1} = x_n - \dfrac{x_n - 2\sin(x_n)}{1 - 2\cos(x_n)}$. Record several iterations to the required accuracy:

    $x_0 = 3$
    $x_1 = 2.08800$
    $x_2 = 1.91223$
    $x_3 = 1.89565$
    $x_4 = 1.89549$
    $x_5 = 1.89549$

    The method converged in five iterations to $x = 1.89549$.

12. For this problem $x_{n+1} = x_n - \dfrac{6x_n^{\,3} - 4x_n + 1}{12x_n - 4}$.

$x_0 = 1.2$
$x_1 = 0.568$
$x_2 = 0.629$
$x_3 = 0.635$
$x_4 = 0.636$
$x_5 = 0.636$

The method converged in five iterations to $x = 0.636$.

13. For this problem $x_{n+1} = x_n - \dfrac{\ln(x_n)\times(\ln(x_n)+4)+1}{\dfrac{2\ln(x_n)+4}{x_n}} = x_n - \dfrac{x_n\ln(x_n)\times(\ln(x_n)+4)+x}{2\ln(x_n)+4}$.

$x_0 = 1$
$x_1 = 0.75$
$x_2 = 0.76489$
$x_3 = 0.76495$
$x_4 = 0.76495$

We estimate the root is at $x = 0.76495$.

14. For this problem $x_{n+1} = x_n - \dfrac{\tan(x_n) - \csc(x_n)}{\sec^2(x_n) + \csc(x_n)\cot(x_n)}$.

$x_0 = 0.7$
$x_1 = 0.899861$
$x_2 = 0.904568$
$x_3 = 0.904557$
$x_4 = 0.904557$

We estimate the root is at $x = 0.904557$.

15. For this problem $x_{n+1} = x_n - \dfrac{\tan^{-1}(x_n) + \cos(x_n)}{\dfrac{1}{1+x^2} - \sin(x)}$.

$x_0 = -2$
$x_1 = -0.6268$
$x_2 = -0.8185$
$x_3 = -0.8165$
$x_4 = -0.8165$

We estimate the root is approximately $x = -0.8165$.