Calculus Concept Collection - Chapter 3

Need for Derivatives: Tangent Lines to a Curve

Answers

1. Secant line.

2. Tangent line

3. $f(x_0+h)-f(x_0)$ is called "the rise" in the process of finding the slope of a tangent line?

4. $(x_0+h)-x_0=h$ is assumed to go to 0 as the two chosen points get closer and closer.

5. The slope of a line tangent to a curve is the limit of the slope of the secant line as the two points getting closer to each other.

6. $y=x-2$

7. $y=-5x+8$

8. $y=-3x+7$

9. $y=3x-8$

10. $y=5x+22$

11. $y=-20x+16$

12. $y=-2x$

13. $y=19x-5$

14. $y=8x+3$

15. $y=10x$

16. $y=-19x-7$

17. $y=x$

18. $y=-2x+3$

19. $y=-3$

20. $y = 36x + 19$

Need for Derivatives: Instantaneous Rates of Change

Answers

Find the Average Rate of Change

1. Average rate of changes is 70.
2. Average rate of changes is 267.
3. Average rate of changes is 248.
4. Average rate of changes is 109.
5. Average rate of changes is -423.

Find the Instantaneous Rate of Change:

6. Instantaneous rate of change of (C) with respect to (x) is -198.
7. Instantaneous rate of change of (N) with respect to (x) is 59.
8. Instantaneous rate of change of (H) with respect to (x) is 80.
9. Instantaneous rate of change of (N)   with respect to (x) is -299.
10. Instantaneous rate of change of (C)   with respect to (x) is -116.

Use the definition of the derivative to find f(x) and then find the equation of the tangent line at x = x0.

11. $f'(x) = 12x$, $(y-54) = 36(x-3)$
12. $f'(x) = \dfrac{1}{2\sqrt{x+2}}$, $(y-\sqrt{10}) = \dfrac{\sqrt{10}}{20}(x-8)$
13. $f'(x) = 9x^2$, $(y+5) = 9(x+1)$
14. $f'(x) = \dfrac{-1}{(x+2)^2}$, $(y-1) = -(x+1)$
15. $f'(x) = 2ax$, $(y-(ab^2-b)) = 2ab(x-b)$

16. $f'(x) = \frac{1}{3x^{2/3}}$, $(y-1) = \frac{1}{3}(x-1)$

17. **Suppose that *f* has the property that *f*(*x* + *y*) = *f*(*x*) + *f*(*y*) + 3*xy* and $\lim_{h\to 0} \frac{f(h)}{h} = 4$. Find *f*(0) and *f*'(*x*).

18. **Find *dy*/*dx*

Solve Rate of Change Problems

19. f'(100) = 9999 is the rate of change of the cost of producing 100 jars.

20. f'(100) = -0.6 means that 100 minutes after the pie is placed in the room, it's temperature is decreasing (slope is negative) at a rate of 0.6 degree per minute.

21. The unit of f'(x) is rate of change of virus per hour , or number of virus per hour?

22. f'(x) is the rate of change per day of the number of people being affected at day x.

23. Average rate of change of J with respect to x is 79.

24. Given $f(x) = 0.008264x^2 - 2x + 196$, then the instantaneous rate of change at x=15 is $f'(15) = 0.016528(15) - 2 = -1.75$ °F /minute. The temperature of the cake is decreasing at 1.75 °F /minute.

The Derivative

Answers

1. $f'(x) = 12x, y = 36x - 54$

2. $f'(x) = \frac{1}{2\sqrt{x+2}}, y = \frac{1}{\sqrt{6}}\left(\frac{1}{2}x + 6\right)$

3. $f'(x) = 9x^2, y = 9x + 4$

4. $f'(x) = \frac{-1}{(x+2)^2}, y = -x$

5. $f'(x) = 2ax, y = 2abx - b(ab+1)$

6. $f'(x) = \frac{1}{3x^{2/3}}, y = \frac{1}{3}x + \frac{2}{3}$

7. 10

8. Hint: Take the limit from both sides.

9. Hint: Take the limit from both sides.

10. $f(0) = 0, f'(x) = 4 + 3x$

11. The definition of a derivative states that

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$$

$$\lim_{h\to 0}\frac{f(x+h) - f(x)}{h} = \lim_{h\to 0}\frac{2(x+h)^2 + (x+h) - (2x^2+2)}{h} = \lim_{h\to 0}\frac{2(x^2+2hx+h^2) + x + h - 2x^2 - x}{h}$$

group like terms:

$$= \lim_{h\to 0}\frac{2x^2 + 4hx + 2h^2 + x + h - 2x^2 - x}{h} = \lim_{h\to 0}\frac{4hx + 2h^2 + h}{h}$$

All terms in the numerator contain a factor of h, so now we may simplify.

$$\lim_{h\to 0}\frac{4hx + 2h^2 + h}{h} = \lim_{h\to 0}\frac{4x + 2h + 1}{1} = \lim_{h\to 0} 4x + 2h + 1$$

Now that *h* is not the denominator, we may safely evaluate the limit.

$$f'(x) = \lim_{h\to 0} 4x + 2h + 1 = 4x + 2(0) + 1 = 4x + 1$$

We wish to know the value of the derivative at x=2.

$f'(2) = 4(2) + 1 = 9$.

12. The definition of a derivative states that

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{3(x+h-1)^2 + 7 - 3(x-1)^2 + 7}{h} = \lim_{h \to 0} \frac{3(x^2 + h^2 + 1 + 2hx - 2x - 2h) + 7 - 3(x^2 - 2x + 1) - 7}{h}$$

Combine like terms.

$$= \lim_{h \to 0} \frac{3(2hx + h^2 - 2h)}{h}$$

Now *h* can be reduced from each term:

$$\lim_{h \to 0} \frac{3(2hx + h^2 - 2h)}{h} = \lim_{h \to 0} \frac{3(2x + h - 2)}{1} = \lim_{h \to 0} 3(2x + h - 2) = \lim_{h \to 0} 6x + 3h - 6$$

and the limit can be evaluated:

$$f'(x) = \lim_{h \to 0} 6x + 3h - 6 = 6x + 3(0) - 6 = 6x - 6$$

13. $f'(x) = \dfrac{1}{2\sqrt{x-1}}$

14. $f'(-5) = \dfrac{-1}{\sqrt{6 - 2(-5)}} = -\dfrac{1}{4}$

15. $f'(x) = -\dfrac{1}{(x+1)^2}$

Differentiation Rules: Constants and Variable Powers

Answers

1. The Power Rule: If n is a real number, then for all real values of x $\frac{d}{dx}[x^n] = nx^{n-1}$.

2. $y' = 35x^6$

3. $y' = -3$

4. $f'(x) = \frac{1}{3}$

5. $y' = 4x^3 - 6x^2 - \frac{5}{2\sqrt{x}}$

6. $y' = 100x^3 - 60x$

7. Given $y'(1) = -4\pi^2(1)^{-4\pi^2-1} = -4\pi^2$

8. $y'(x) = 0$

9. $u'(2) = -5\pi^3(2)^{-5\pi^3-1} = -1.7 \times 10^{-45}$

10. $y'(4) = 0$

11. $d'(1) = -0.37(1)^{-1.37} = -0.37$

12. $g'(x) = -3x^{-4}$

13. $u'(x) = 0.096x^{-0.004}$

14. $k'(x) = 1$

15. $y = -5\pi^3 x^{-5\pi^3-1}$

Differentiation Rules: Sums and Differences

Answers

1. $y' = \frac{1}{2}(3x^2) - \frac{1}{2}(4x) + \frac{1}{2} \cdot 0 = \frac{3}{2}x^2 - 2x$

2. $y' = 3\sqrt{2}x^2 - \frac{2}{\sqrt{2}}x + 2$

3. $y' = 0 - 0 + 2x - 0 - 0 + 1 = 2x + 1$

4. $y' = -3x^{-4} + \frac{-7}{x^8}$

5. $y' = \frac{1}{2\sqrt{x}} - \frac{1}{2x^{3/2}} = \frac{1}{2\sqrt{x}}\left(1 - \frac{1}{x}\right)$

6. $\frac{d}{dx}(9x^2 - 24x + 16) = 18x - 24$

7. $f'(x) = -9.3x^9 - \pi^{-5/4}\frac{5}{12}x^{\frac{-17}{12}}$

8. $\frac{d}{dx}(2x+1)^2 = \frac{d}{dx}(4x^2 + 4x + 1) = 8x + 4$

9. $\frac{d}{dx}(25x^2 - 30x + 9) = 50x - 30$

10. $v'(0) = -9(0)^2 + 10(0) - 2 = -2$

11. Just differentiate each term individually.

$$f'(x) = 2 \times 2x^{2-1} + 3 + 0 = 4x + 3$$

12. Rewrite the x's in the denominators as negative powers and roots as fractional powers, then differentiate term-by-term.

$$f(x) = \frac{1}{\sqrt{x}} - \frac{1}{x} = x^{-1/2} - x^{-1}$$

$$f'(x) = -\frac{1}{2}x^{-3/2} - (-1)x^{-2} = -\frac{1}{2}x^{-3/2} + x^{-2} = -\frac{1}{2x^{3/2}} + \frac{1}{x^2}$$

$$f'(1) = -\frac{1}{2(1)^{3/2}} + \frac{1}{1^2} = -\frac{1}{2} + 1 = \frac{1}{2}.$$

13. First determine the product of the binomial, then differentiate term-by-term:

$$y = (x+1)(x+2) = x^2 + x + 2x + 2 = x^2 + 3x + 2$$

$$\frac{dy}{dx} = 2x^1 + 3 + 0 = 2x + 3$$

at $x = -\frac{1}{2}$,

$$\frac{dy}{dx} = 2(-\frac{1}{2}) + 3 = -1 + 3 = 2.$$

14. First find a general form for *f′*.

$$f'(x) = 2a \times 3x^2 + 2x^1 = 6ax^2 + 2x$$

$$f'(-2) = 6a(-2)^2 + 2(-2) = 24a - 4$$

$$24a - 4 = 0$$

$$24a = 4$$

$$a = \frac{1}{6}$$

15. First, distribute a. Then differentiate term-by-term.

$$f(x) = a(x^2 - 5) = ax^2 - 5a$$

$$f'(x) = 2ax + 0$$

$$f''(x) = 2a$$

$$f''(5) = 20 = 2a \rightarrow a = 10$$

Differentiation Rules: Products and Quotients

Answers

1. Write y as the product of functions: $y = f(x) \times g(x)$,

   $f(x) = x^3 - 3x^2 + x \rightarrow f'(x) = 3x^2 - 6x + 1$

   $g(x) = 2x^3 + 7x^4 \rightarrow g'(x) = 6x^2 + 28x^3$

   By the product rule, $\frac{dy}{dx} = f(x)g'(x) + g(x)f'(x)$:

   $\frac{dy}{dx} = (x^3 - 3x^2 + x)(6x^2 + 28x^3) + (3x^2 - 6x + 1)(2x^3 + 7x^4)$

2. $f(x) = \left(\frac{1}{x} + \frac{1}{x^2}\right) \rightarrow f'(x) = \frac{-1}{x^2} - \frac{2}{x^3}$, and

   $g(x) = (3x^4 - 7) \rightarrow g'(x) = 12x^3$

   By the product rule: $\frac{dy}{dx} = \left(\frac{1}{x} + \frac{1}{x^2}\right)12x^3 + \left(\frac{-1}{x^2} - \frac{2}{x^3}\right)(3x^4 - 7)$

3. $f(x) = (-3x^2 + x + 4) \rightarrow f'(x) = (-6x + 1)$, and

   $g(x) = (-3x - 3) \rightarrow g'(x) = -3$

   By the product rule: $\frac{dy}{dx} = (-3x^2 + x + 4)(-3) + (-6x + 1)(-3x - 3)$

4. By the product rule: $(k \cdot r)'(-2) = k(-2)r'(-2) + r(-2)k'(-2) = 54$, then

   $r(-2) = \frac{54 - k(-2)r'(-2)}{k'(-2)} = \frac{54}{18} = 3$

5. By the product rule: $g'(x) = (4x^2 - 4x - 5)(3) + (8x - 4)(3x - 3)$, then

   $g'(2) = (16 - 8 - 5)(3) + (16 - 4)(6 - 3) = 45$

6. $\frac{dy}{dx} = (x^2 - 3)(-4x + 4) + (2x)(-2x^2 + 4x - 1)$

7. By the product rule: $(ta)' = ta' + t'a$, then

   $(ta)'(1) = t(1)a'(1) + t'(1)a(1) = 272$

$$a(1)=\frac{272-t(1)a'(1)}{t'(1)}=\frac{272}{17}=16$$

8. By the product rule: $d'(x)=(2x^2+3x-1)(2)+(4x+3)(2x+1)$, then
$d'(-1)=(2-3-1)(2)+(-4+3)(-2+1)=-3$.

9. Write y as the product of functions: $y=f(x)\times g(x)$,

$$f(x)=x^3+2x+1\rightarrow f'(x)=3x^2+2$$

$$g(x)=4x^4-7x-8\rightarrow g'(x)=16x^3-7$$

By the product rule,

$$\frac{dy}{dx}=f'(x)g(x)+g'(x)f(x)=(3x^2+2)(4x^4-7x-8)+(16x^3-7)(x^3+2x+1)$$

10. Write y as the product of functions, $y=f(x)\times g(x)$:

$$y=\frac{x^2+2}{\sqrt{4x}}=(x^2+2)\times\frac{1}{\sqrt{4x}}=(x^2+2)\times\frac{1}{2\sqrt{x}}=(x^2+2)\times\frac{1}{2}x^{-1/2}$$

$$f(x)=x^2+2\rightarrow f'(x)=2x$$

$$g(x)=\frac{1}{2}x^{-1/2}\rightarrow g'(x)=-\frac{1}{4}x^{-3/2}$$

$$y'=f'(x)g(x)+g'(x)f(x)=2x(\frac{1}{2}x^{-1/2})+(-\frac{1}{4}x^{-3/2})(x^2+2)=\frac{2x}{2\sqrt{x}}-\frac{x^2+2}{4x^{3/2}}=\sqrt{x}-\frac{x^2+2}{4x^{3/2}}$$

11. First find the derivative using the product rule.

$$y=f(x)\times g(x)=(2x^5+x^4-x^3-x^2-x)\times(2x^5+x^4-x^3-x^2-x)$$

$$f(x)=g(x)=2x^5+x^4-x^3-x^2-x\rightarrow f'(x)=g'(x)=10x^4+4x^3-3x^2-2x-1$$

$$m_{tan}=f'(x)g(x)+g'(x)f(x)=f'(x)f(x)+f'(x)f(x)=2f'(x)f(x)$$

$$=2(10x^4+4x^3-3x^2-2x-1)(2x^5+x^4-x^3-x^2-x)$$

Evaluate the slope at x=-1:

$$m_{tan}=2\times(10(-1)^4+4(-1)^3-3(-1)^2-2(-1)-1)\times(2(-1)^5+(-1)^4-(-1)^3-(-1)^2-(-1))$$

$$=2\times(10-4-3+2-1)\times(-2+1+1-1+1)=2\times4\times0=0$$

$m_{tan}=0$ at x=-1. Next we need the y-coordinate:

$$y=(2(-1)^5+(-1)^4-(-1)^3-(-1)^2-(-1))^2=(-2+1+1-1+1)^2=0$$

Since the slope is 0, the tangent line is the horizontal line $y=0$.

12. First write y as the product of functions, $y = f(x) \times g(x)$.

$$y = \frac{2x^3 - 1}{x} = (2x^3 - 1) \times \frac{1}{x} = (2x^3 - 1) \times x^{-1}$$

$$f(x) = 2x^3 - 1 \rightarrow f'(x) = 6x^2$$

$$g(x) = x^{-1} \rightarrow g'(x) = -x^{-2} = -\frac{1}{x^2}$$

$$y' = f'(x)g(x) + g'(x)f(x) = 6x^2 \times x^{-1} + (-x^{-2}) \times (2x^3 - 1) = \frac{6x^2}{x} - \frac{2x^3 - 1}{x^2} = 6x - \frac{2x^3 - 1}{x^2}$$

13. There are various approaches. For instance, one may FOIL the first two terms of the product:

$$y = (2x+7)(x^2 - 10)(3x^2 + 3x + 5) = (2x^3 + 7x^2 - 20x - 70) \times (3x^2 + 3x + 5)$$

$$f(x) = 2x^3 + 7x^2 - 20x - 70 \rightarrow f'(x) = 6x^2 + 14x - 20$$

$$g(x) = 3x^2 + 3x + 5 \rightarrow g'(x) = 6x + 3$$

$$\frac{dy}{dx} = f'(x)g(x) + g'(x)f(x) = (6x^2 + 14x - 20)(3x^2 + 3x + 5) + (6x+3)(2x^3 + 7x^2 - 20x - 70)$$

14. Let $h(x) = \left(\frac{f(x)}{g(x)} \right)$ with $g(x) \neq 0$. Then $f(x) = h(x) \cdot g(x)$.

By the product rule: $f'(x) = h \cdot g' + h'g = \frac{f}{g} g' + h'g$.

Solving for h' : $h' = \frac{f' - \frac{f}{g} g'}{g} = \frac{\frac{gf'}{g} - \frac{fg'}{g}}{g} = \frac{gf' - fg'}{g^2}$

15. By the quotient rule: $\left(\frac{u}{q} \right)' = \frac{qu' - uq'}{q^2}$.

Then $\left(\frac{u}{q} \right)'(0) = \frac{q(0)u'(0) - u(0)q'(0)}{q^2(0)} = \frac{q(0)98}{q^2(0)} = 7$, and

$q(0) = \frac{98}{7} = 14$.

16. Let $b(x) = \frac{f(x)}{g(x)}$; then

$f(x)=x^2-5x+4 \to f(2)=-2$
$f'(x)=2x-5 \to f'(2)=-1$ with
$g(x)=-5x+2 \to g(2)=-10$
$g'(x)=-5 \to g'(2)=-5$

$$b'(2)=\frac{-10(-1)-(-2)(-5)}{(-10)^2}=0$$

17. Start by finding $\frac{dy}{dx}$.

$$y=\frac{x+1}{2x-1}=\frac{f(x)}{g(x)}$$

$f(x)=x+1 \to f'(x)=1$

$g(x)=2x-1 \to g'(x)=2$

$$\frac{dy}{dx}=\frac{f'(x)g(x)-g'(x)f(x)}{(g(x))^2}=\frac{1(2x-1)-2(x+1)}{(2x-1)^2}=\frac{2x-1-2x-1}{(2x-1)^2}=\frac{-2}{(2x-1)^2}$$

Use the quotient rule again on $\frac{dy}{dx}$:

$f(x)=-2 \to f'(x)=0$

$g(x)=(2x-1)^2=4x^2-4x+1 \to g'(x)=8x-4$

$$\frac{d^2y}{dx^2}=\frac{f'(x)g(x)-g'(x)f(x)}{(g(x))^2}=\frac{0(2x-1)^2-(8x-4)(-2)}{(2x-1)^4}=\frac{2(8x-4)}{(2x-1)^4}=\frac{8(2x-1)}{(2x-1)^4}=\frac{8}{(2x-1)^3}$$

$$\frac{d^2y}{dx^2}=\frac{8}{(2x-1)^3}$$

18. $y=\frac{x^3-3}{x^2+1}=\frac{f(x)}{g(x)}$

$f(x)=x^3-3 \to f'(x)=3x^2$

$g(x)=x^2+1 \to g'(x)=2x$

$$y'=\frac{f'(x)g(x)-g'(x)f(x)}{(g(x))^2}=\frac{3x^2(x^2+1)-2x(x^3-3)}{(x^2+1)^2}=\frac{3x^4+3x^2-2x^4+6x}{(x^2+1)^2}=\frac{x^4+3x^2+6x}{(x^2+1)^2}$$

$$y'=\frac{x^4+3x^2+6x}{(x^2+1)^2}$$

19. The function can be simplified.

$y=\frac{x^2+6x+5}{x+5}=\frac{(x+1)(x+5)}{x+5}=x+1$ wherever it is defined.

$$\frac{dy}{dx}=1.$$

20. $y=\frac{x^2(x-3)}{x^5-5}=\frac{x^3-3x^2}{x^5-5}=\frac{f(x)}{g(x)}$

$$f(x)=x^3-3x^2 \to f'(x)=3x^2-6x$$

$$g(x)=x^5-5 \to g'(x)=5x^4$$

$$y'=\frac{f'(x)g(x)-g'(x)f(x)}{(g(x))^2}=\frac{(3x^2-6x)(x^5-5)-5x^4(x^3-3x^2)}{(x^5-5)^2}.$$

At $x=-1$,

$$y'=\frac{(3(-1)^2-6(-1))((-1)^5-(-1))-5(-1)^4((-1)^3-3(-1)^2)}{((-1)^5-5)^2}=\frac{(3+6)(0)-5(1)(-1-3)}{(-1-5)^2}$$

$$=\frac{-5\times-4}{(-6)^2}=\frac{-20}{36}=-\frac{5}{8}$$

21. Since the denominator is a constant, it is not necessary to use the quotient rule.

$$f(x)=\frac{x^5-3x^3-1}{4}=\frac{1}{4}(x^5-3x^3-1)$$

$$f'(x)=\frac{1}{4}(5x^4-9x^2)$$

$$f''(x)=\frac{1}{4}(20x^3-18x)=5x^3-\frac{9}{2}x$$

Derivatives of Trigonometric Functions

Answers

1. $y' = x\cos x + \sin x$

2. $y' = 2x\cos x - x^2\sin x - \tan x - x\sec^2 x$

3. $y' = 2\cos x\sin x$

4. $y' = \dfrac{2\cos x}{(\sin x + 1)^2}$

5. $y' = 1 + \left(\dfrac{\tan x + 1}{1 - \tan x}\right)^2$

6. $y' = \dfrac{\cot}{2\sqrt{x}} - \sqrt{x}\csc^2 x$

7. $y' = 1$

8. $y' = \sec^2 x$

9. $y'(\pi/6) = -2\sqrt{3}$

10. $(x^5\cos(x))' = 5x^4\cos(x) - x^5\sin(x)$

11. By the product rule

$$(x^2\csc(x))' = (x^2)'\csc(x) + x^2\csc(x)' = 2x\csc(x) - x^2(\csc(x)\cot(x))$$

12. The derivative is equal to $\sec(x)\tan(x)$.

To obtain this, note that the expression can be simplified as follows

$$\csc(x)\tan(x) = \frac{1}{\sin(x)}\frac{\sin(x)}{\cos(x)} = \frac{1}{\cos(x)} = \sec(x)$$

And the derivative of secant is just $\sec(x)\tan(x)$.

13. $(\sec(x))' = (\dfrac{1}{\cos(x)})' = \dfrac{0-(\cos(x))'}{\cos^2(x)} = \dfrac{\sin(x)}{\cos^2(x)} = \dfrac{\sin(x)}{\cos(x)}\dfrac{1}{\cos(x)} = \tan(x)\sec(x)$

14. The solution is $\sec^2(x)$.

We first simplify the expression:

$$\cot(\frac{\pi}{2}-x)=\frac{1}{\tan(\frac{\pi}{2}-x)}=\frac{1}{-\tan(x-\frac{\pi}{2})}=-\frac{\cos(x-\frac{\pi}{2})}{\sin(x-\frac{\pi}{2})}=\frac{\cos(x)}{\sin(x)}=\tan(x)$$

And then the derivative of tangent is just secant squared.

15. The derivative equals 0.

We can simplify the expression:

$$\csc^2(x)-\cot^2(x)=\frac{1}{\sin^2(x)}-\frac{\cos^2(x)}{\sin^2(x)}=\frac{1-\cos^2(x)}{\sin^2(x)}=\frac{\sin^2(x)}{\sin^2(x)}=1$$

This makes it clear that the derivative is zero.

This problem is also solvable by taking the derivative of the expression as it is written in the problem, but that method is longer and has more room for error.

Differentiation Rules: The Chain Rule

Answers

1. $f'(x) = 39(2x^2 - 3x)^{38}(4x - 3)$

2. $f'(x) = -3\left(\dfrac{x^2}{x^5 - 5}\right)^4 \left(\dfrac{3x^5 - 10}{x^3}\right)$

3. $f'(x) = \dfrac{-3(x-1)}{\sqrt{(3x^2 - 6x + 1)}^3}$

4. $f'(x) = 3\sin^2 x \cos x$

5. $f'(x) = 3x^2 \cos x^3$

6. $f'(x) = 9x^2 \cos x^3 \sin^2 x^3$

7. $f'(x) = 20x^4 \sec^2(4x^5)$

8. $f'(x) = \dfrac{2(1 - \sin 2x \ \cos x)}{\sqrt{4x - \sin^2 2x}}$

9. $f'(x) = \dfrac{\cos(3x-2)\cos x + 3\sin(3x-2)\sin x}{\cos^2(3x-2)}$

10. $f'(x) = 13(5x+8)^3(x^3+7x)^{12}(3x^2+7) + 15(x^3+7x)^{13}(5x+8)^2$

11. $f'(x) = \dfrac{3(x-3)^2}{(2x-5)^3}$

12. Let $u = 3x^2 - 1$. By the chain rule,

$$\frac{d}{dx}[f(u)] = f'(u)\frac{du}{dx}$$

where $f(u) = 5\cos u$. Thus

$$\frac{dy}{dx} = 5(-\sin u).(6x)$$
$$= -5\sin u.(6x)$$
$$= -30x\sin(3x^2 - 1)$$

13. $\frac{1}{2}(x^3 + x^5 + 89)^{-\frac{1}{2}} \times (3x^2 + 5x^4)$.

The outer function is the square root, whose derivative is $\frac{1}{2}x^{-\frac{1}{2}}$. Substituting in the inner function gives $\frac{1}{2}(x^3 + x^5 + 89)^{-\frac{1}{2}}$. Multiplying by the derivative of the inner function gives our answer: $\frac{1}{2}(x^3 + x^5 + 89)^{-\frac{1}{2}} \times (3x^2 + 5x^4)$.

14. $\cos(\sin(\sin(x))\cos(\sin(x)\cos(x)$.

To solve this problem we must apply the chain rule multiple times. The derivative of the outer sine is cosine, and composing that with sine of sine of x yields $\cos(\sin(\sin(x))$. We must multiply this by the derivative of the interior function, sine of sine of x. Using the chain rule again, we get:
$\frac{d}{dx}\sin(\sin(x)) = \cos(\sin(x))\cos(x)$. Multiplying yields:
$\cos(\sin(\sin(x))\cos(\sin(x)\cos(x)$.

15. The rule that is proven is the inverse function theorem.

Our equation is $f(f^{-1}(x)) = x$.

We differentiate the left hand side using the chain rule.

$$\frac{d}{dx}f(f^{-1}(x)) = f'(f^{-1}(x))\frac{d}{dx}f^{-1}(x).$$

The right hand side is also easy to differentiate:

$$\frac{d}{dx}x = 1$$

Therefore, since $f(f^{-1}(x)) = x$, we have that:

$$f'(f^{-1}(x))\frac{d}{dx}f^{-1}(x) = 1.$$

Solving algebraically for the derivative of inverse, we have:

$$\frac{d}{dx}f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}$$

This is the inverse function theorem.

Derivatives of Exponential Functions

Answers

1. $y' = 7^x \ln 7$

2. $y' = 2\ln 3 \cdot 3^{2x}$

3. $y' = 5^x \ln 5 - 6x$

4. $y' = 2^{x^2+1} \cdot x \cdot \ln 2.$

5. $\frac{d}{dx}[e^u] = u'e^u \Rightarrow y' = 2xe^{x^2}.$

6. $f'(x) = -\frac{2\alpha k(x-x_0)}{\sqrt{\pi\sigma}} e^{-\alpha k(x-x_0)^2}$

7. $y' = 6e^{6x}$

8. $y' = (9x^2 - 4x)e^{3x^3-2x^2+6}$

9. $y' = \frac{4}{(e^x+e^{-x})^2}$

10. $y' = -e^x \cot e^x$

11. $y' = -e^{-x}3^x + e^{-x}3^x \ln 3 = -e^{-x}3^x(1-\ln 3)$

12. $y' = -2\ln 3(x+1)3^{-x^2+2x+1}$

13. $y' = 2^x 3^x \ln 2 + 2^x 3^x \ln 3 = 2^x 3^x (\ln 2 + \ln 3)$

14. $y = -e^{-x}\sin x + e^{-x}\cos x = -e^{-x}(\sin x - \cos x)$

15. $f'(x) = 3x^2 + 2e^x$, so the slope at (0,2) is $e^0 = 1$, and the line is $y = x + 2$.

Implicit Differentiation

Answers

1. $y' = \frac{-x}{y}$

2. $y' = \frac{-y(2x+3)}{x(x+3)}$

3. $y' = \frac{-y^2}{x^2}$

4. $y' = \sqrt{\frac{y}{x}}$

5. $y' = \frac{1 - 25y^2\cos(25xy^2)}{50xy\cos(25xy^2)}$

6. $y' = \frac{x}{y}$

7. 1

8. $\frac{-1}{1+\pi}$

9. $y' = \frac{2y}{x^2}$

10. Implicit differentiation yields: $y' = \frac{k}{2y}$, so that $y'(x_0) = \frac{k}{2y_0}$, which is the slope of the tangent line at (x_0, y_0). The equation of the tangent line is $y - y_0 = \frac{k}{2y_0}(x - x_0)$. Multiplying both sides by y_0, using the fact that $y_0{}^2 = kx_0$, and simplifying algebraically gives the result $y_0 y = \frac{k}{2}(x + x_0)$.

11. $\frac{d}{dx}(x\sin(y) + y\sin(x)) = \frac{d}{dx}(x\sin(y)) + \frac{d}{dx}(y\sin(x))$.

Use the product rule on each term in the sum.

$$\frac{d}{dx}(x\sin(y)) = \frac{d}{dx}(x)\times\sin(y) + x\times\frac{d}{dx}(\sin(y)) = 1\times\sin(y) + x\times\cos(y)y'$$

$$\frac{d}{dx}(y\sin(x)) = \frac{d}{dx}y\times\sin(x) + y\times\frac{d}{dx}\sin(x) = y'\sin x + y\cos(x)$$

Sum the two terms' derivatives.

$$\frac{d}{dx}(x\sin(y) + y\sin(x)) = \sin(y) + xy'\cos(y) + y'\sin(x) + y\cos(x)\,.$$

12. Differentiate term-by-term.

$$2x + xy' + 1y + 2yy' = 0\,.$$

Move terms that do not contain *y'* to the RHS.

$$xy' + 2yy' = -2x - y$$

Isolate *y'*.

$$(x+2y)y' = -2x - y$$

$$y' = \frac{-2x-y}{x+2y}$$

13. First, differentiate term-by-term

$$3y^2y' + 2y^2 + 2x(2y)y' - 1 = 0$$

We could solve for *y'* now, but since we are only interested in evaluating *y'* at one point, it is probably easier to plug in (*x*,*y*)=(1,1) and then solve for *y'*.

$$3(1)^2y' + 2(1)^2 + 2(1)(2(1))y' - 1 = 0$$

$$3y' + 2 + 4y' - 1 = 0$$

$$7y' + 1 = 0$$

$$7y' = -1$$

$$y' = -\frac{1}{7}$$

This gives the slope of the tangent line at (1,1). Now write the equation of the line in point-slope form.

$$y = 1 - \frac{1}{7}(x-1)$$

14. Perform the chain rule, using the product rule to find the derivative of the "inside function":

$$\frac{d}{dx}(\sqrt{xy}) = \frac{d}{dx}(xy)^{1/2} = \frac{1}{2}(xy)^{-1/2}\times\frac{d}{dx}(xy) = \frac{1}{2}(xy)^{-1/2}\times(1y + xy') = \frac{y+xy'}{2\sqrt{xy}}\,.$$

15. First find the first derivative by differentiating term-by-term.

$$2y\frac{dy}{dx}+\cos(y)\frac{dy}{dx}=1$$

$$\frac{dy}{dx}(2y+\cos(y))=1$$

$$\frac{dy}{dx}=\frac{1}{2y+\cos(y)}=(2y+\cos(y))^{-1}$$

Find $\frac{d^2y}{dx^2}$ by differentiating again; we can use quotient rule or chain rule. Here are the steps for chain rule:

$$\frac{d^2y}{dx^2}=\frac{d}{dx}(2y+\cos(y))^{-1}=-1\times(2y+\cos(y))^{-2}\times\frac{d}{dx}(2y+\cos(y))$$

$$=-\frac{1}{(2y+\cos(y))^2}\times(2\frac{dy}{dx}-\sin(y)\times\frac{dy}{dx})=-\frac{1}{(2y+\cos(y))^2}\times(2-\sin(y))\times\frac{dy}{dx}$$

$$=-\frac{2-\sin(y)}{(2y+\cos(y))^2}\times\frac{dy}{dx}$$

Substitute in $\frac{dy}{dx}=\frac{1}{2y+\cos(y)}$

$$\frac{d^2y}{dx^2}=-\frac{2-\sin(y)}{(2y+\cos(y))^2}\times\frac{1}{2y+\cos(y)}=-\frac{2-\sin(y)}{(2y+\cos(y))^3}$$

Higher Order Derivatives

Answers

1. Given: $v(x) = -4x^3 + 3x^2 + 2x + 3$

$v'(x) = -12x^2 + 6x + 2$

$v''(x) = -24x + 6$

2. Given: $m(x) = x^2 + 5x$. What is $m''(x)$?

$m'(x) = 2x + 5$

$m''(x) = 2$

3. Given: $d(x) = 3x^4 e^x$. What is $d''(x)$?

$d'(x) = 3x^4 e^x + 12x^3 e^x$

$d''(x) = 3x^4 e^x + 12x^3 e^x + 12x^3 e^x + 36x^2 e^x$

$d''(x) = 3x^2 e^x (x^2 + 8x + 12)$

4. Given: $t(x) = -2x^5 sin(x)$

$t'(x) = -2x^5 \cos(x) - 10x^4 sin(x) = -2x^4(\cos(x) + 5\sin(x))$

$t''(x) = -2x^4(-\sin(x) + 5\cos(x)) - 8x^3(\cos(x) + 5\sin(x))$

$t''(x) = 2x^3[(x-20)s\,\text{in}(x) - (5x+4)\cos(x)]$

5. Given: $y = 3x^5 e^x$

$y' = 3x^5 e^x + 15x^4 e^x$

$y'' = 3x^5 e^x + 15x^4 e^x + 15x^4 e^x + 60x^3 e^x$

$y'' = 3x^3 e^x (x^2 + 10x + 20)$

6. Given $y = \frac{2}{x^3}$, then $y' = \frac{-6}{x^4}$; $y'' = \frac{24}{x^5}$; and $y''' = \frac{-120}{x^6}$.

$y'''(1) = \frac{-120}{1^6} = -120$

7. By the quotient rule: $\left(\frac{u}{q}\right)' = \frac{qu' - uq'}{q^2}$.

Then $\left(\frac{u}{q}\right)'(0) = \frac{q(0)u'(0) - u(0)q'(0)}{q^2(0)} = \frac{q(0)98}{q^2(0)} = 7$, and

$q(0) = \frac{98}{7} = 14$.

8. Let $b(x) = \frac{f(x)}{g(x)}$; then

$$f(x) = x^2 - 5x + 4 \rightarrow f(2) = -2$$
$$f'(x) = 2x - 5 \rightarrow f'(2) = -1 \text{ with}$$
$$g(x) = -5x + 2 \rightarrow g(2) = -10$$
$$g'(x) = -5 \rightarrow g'(2) = -5$$
$$b'(2) = \frac{-10(-1) - (-2)(-5)}{(-10)^2} = 0$$

9. Given: $m(x) = \frac{e^x}{3x+4}$

$$m'(x) = \frac{(3x+4)e^x - e^x(3)}{(3x+4)^2} = \frac{(3x+1)e^x}{(3x+4)^2}$$

10. Given: $y = \frac{sin(x)}{x-4}$

$$y' = \frac{(x-4)cos(x) - \sin x}{(x-4)^2}$$

11. Given $q(x) = \frac{x}{sin(x)}$.

$$q'(x) = \frac{\sin x - x\cos x}{sin(x)^2}$$

$$q''(x) = \frac{sin(x)^2(x\sin x) - 2\sin x\cos x(\sin x - x\cos x)}{sin(x)^4}$$

12.

$$f(t) = t^3 + t$$
$$f'(t) = 3t^2 + 1$$
$$f''(t) = 6t$$

This last expression is our acceleration.

13.

$$f(t) = \sin(t) + 3t^2$$
$$f'(t) = \cos(t) + 6t$$
$$f''(t) = -\sin(t) + 6$$

Since the sine function is bounded by one, the car is accelerating.

14. Negative jerk.

$$f(t) = \cos(-t)$$
$$f'(t) = \sin(-t)$$
$$f''(t) = -\cos(-t)$$
$$f'''(t) = -\sin(-t)$$

At the prescribed time, we have that $-\sin(-\frac{3\pi}{2}) = \sin(\frac{3\pi}{2}) = -1$.

15. First we take derivatives.

$$f(t) = \frac{1}{12}t^4 - \frac{3}{6}t^3 - 5t^2 + \pi^\pi$$
$$f'(t) = \frac{1}{3}t^3 - \frac{3}{2}t^2 - 10t$$
$$f''(t) = t^2 - 3t + 10$$

Now we know that $t^2 - 3t - 10 = (t-5)(t+2)$. Assuming time begins at zero, the only point at which there is no acceleration is $t = 5$.

16. N+1. Any polynomial, differentiated enough times, will be zero. Each term will become zero when the number of times its been differentiated exceeds its degree, i.e. x^2 differentiated three times is zero, x^3 differentiated four times is zero, etc. Therefore an N-degree polynomial's derivative will become zero after it is differentiated N+1 times.