Additive and Multiplicative Rules for Probability

Raja Almukkahal
Larry Ottman
Danielle DeLancey
Addie Evans
Ellen Lawsky
Brenda Meery

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Additive and Multiplicative Rules for Probability

• Calculate probabilities using the Additive Rule.
• Calculate probabilities using the Multiplicative Rule.
• Identify events that are not mutually exclusive and explain how to represent them in a Venn diagram.
• Understand the condition of independence.

The Additive and Multiplicative Rules

Venn Diagrams

When the probabilities of certain events are known, we can use these probabilities to calculate the probabilities of their respective unions and intersections. We use two rules, the Additive Rule and the Multiplicative Rule, to find these probabilities. The examples that follow will illustrate how we can do this.

Illustrating the Additive Rule of Probability

Suppose we have a loaded (unfair) die, and we toss it several times and record the outcomes. We will define the following events:

\[ A : \text{observe an even number} \]

\[ B : \text{observe a number less than 3} \]

Let us suppose that we have \( P(A) = 0.4, \ P(B) = 0.3, \) and \( P(A \cap B) = 0.1. \) We want to find \( P(A \cup B). \)

It is probably best to draw a Venn diagram to illustrate this situation. As you can see, the probability of events \( A \) or \( B \) occurring is the union of the individual probabilities of each event.
Therefore, adding the probabilities together, we get the following:

\[ P(A \cup B) = P(1) + P(2) + P(4) + P(6) \]

We have also previously determined the probabilities below:

\[ P(A) = P(2) + P(4) + P(6) = 0.4 \]
\[ P(B) = P(1) + P(2) = 0.3 \]
\[ P(A \cap B) = P(2) = 0.1 \]

If we add the probabilities \( P(A) \) and \( P(B) \), we get:

\[ P(A) + P(B) = P(2) + P(4) + P(6) + P(1) + P(2) \]

Note that \( P(2) \) is included twice. We need to be sure not to double-count this probability. Also note that 2 is in the intersection of \( A \) and \( B \). It is where the two sets overlap. This leads us to the following:

\[ P(A \cup B) = P(1) + P(2) + P(4) + P(6) \]
\[ P(A) = P(2) + P(4) + P(6) \]
\[ P(B) = P(1) + P(2) \]
\[ P(A \cap B) = P(2) \]
\[ P(A \cup B) = P(A) + P(B) - P(A \cap B) \]

This is the **Additive Rule of Probability**, which is demonstrated below:

\[ P(A \cup B) = 0.4 + 0.3 - 0.1 = 0.6 \]

What we have shown is that the probability of the union of two events, \( A \) and \( B \), can be obtained by adding the individual probabilities, \( P(A) \) and \( P(B) \), and subtracting the probability of their intersection (or overlap), \( P(A \cap B) \). The Venn diagram above illustrates this union.
The Additive Rule of Probability

The probability of the union of two events can be obtained by adding the individual probabilities and subtracting the probability of their intersection: 
\[ P(A \cup B) = P(A) + P(B) - P(A \cap B). \]

We can rephrase the definition as follows: The probability that either event \( A \) or event \( B \) occurs is equal to the probability that event \( A \) occurs plus the probability that event \( B \) occurs minus the probability that both occur.

Using the Additive Rule of Probability

1. Consider the experiment of randomly selecting a card from a deck of 52 playing cards. What is the probability that the card selected is either a spade or a face card?

Our event is defined as follows:

\[ E : \text{card selected is either a spade or a face card} \]

There are 13 spades and 12 face cards, and of the 12 face cards, 3 are spades. Therefore, the number of cards that are either a spade or a face card or both is \( 13 + 9 = 22 \). That is, event \( E \) occurs when 1 of 22 cards is selected, the 22 cards being the 13 spade cards and the 9 face cards that are not spade. To find \( P(E) \), we use the Additive Rule of Probability. First, define two events as follows:

\[ C : \text{card selected is a spade} \]

\[ D : \text{card selected is a face card} \]

Note that \( P(E) = P(C \cup D) = P(C) + P(D) - P(C \cap D) \). Remember, with event \( C \), 1 of 13 cards that are spades can be selected, and with event \( D \), 1 of 12 face cards can be selected. Event \( C \cap D \) occurs when 1 of the 3 face card spades is selected. These cards are the king, jack, and queen of spades. Using the Additive Rule of Probability formula:

\[
P(A \cup B) = P(A) + P(B) - P(A \cap B) \\
= \frac{13}{52} + \frac{12}{52} - \frac{3}{52} \\
= 0.250 + 0.231 - 0.058 \\
= 0.423 \\
= 42.3%\]
Recall that we are subtracting 0.058 because we do not want to double-count the cards that are at the same time spades and face cards.

2. If you know that 84.2% of the people arrested in the mid 1990’s were males, 18.3% of those arrested were under the age of 18, and 14.1% were males under the age of 18, what is the probability that a person selected at random from all those arrested is either male or under the age of 18?

First, define the events:

\[ A : \text{person selected is male} \]

\[ B : \text{person selected in under 18} \]

Also, keep in mind that the following probabilities have been given to us:

\[
P(A) = 0.842 \\
P(B) = 0.183 \\
P(A \cap B) = 0.141
\]

Therefore, the probability of the person selected being male or under 18 is \( P(A \cup B) \) and is calculated as follows:

\[
P(A \cup B) = P(A) + P(B) - P(A \cap B) \\
= 0.842 + 0.183 - 0.141 \\
= 0.884 \\
= 88.4\%
\]

This means that 88.4% of the people arrested in the mid 1990’s were either male or under 18.

**Mutually Exclusive Events**

If \( A \cap B \) is empty \((A \cap B = \varnothing)\), or, in other words, if there is not overlap between the two sets, we say that \( A \) and \( B \) are **mutually exclusive**.

The figure below is a Venn diagram of mutually exclusive events. For example, set \( A \) might represent all the outcomes of drawing a card, and set \( B \) might represent all the outcomes of tossing three coins. These two sets have no elements in common.
If the events $A$ and $B$ are mutually exclusive, then the probability of the union of $A$ and $B$ is the sum of the probabilities of $A$ and $B$: $\Pr(A \cup B) = \Pr(A) + \Pr(B)$. Note that since the two events are mutually exclusive, there is no double-counting.

### Calculating Probability Using the Additive Rule

If two coins are tossed, what is the probability of observing at least one head?
First, define the events as follows:

- $A$: observe only one head
- $B$: observe two heads

Now the probability of observing at least one head can be calculated as shown:

$$
\Pr(A \cup B) = \Pr(A) + \Pr(B) = 0.5 + 0.25 = 0.75 = 75\% 
$$

### The Multiplicative Rule of Probability

Recall from the previous section that conditional probability is used to compute the probability of an event, given that another event has already occurred:

$$
\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)}
$$

This can be rewritten as $\Pr(A \cap B) = \Pr(A \mid B) \cdot \Pr(B)$ and is known as the **Multiplicative Rule of Probability**. The Multiplicative Rule of Probability says that the probability that both $A$ and $B$ occur equals the probability that $B$ occurs times the conditional probability that $A$ occurs, given that $B$ has occurred.

### Using the Multiplicative Rule of Probability

1. In a certain city in the USA some time ago, 30.7% of all employed female workers were white-collar workers. If 10.3% of all workers employed at the city government were female, what is the probability that a randomly selected employed worker would have been a female white-collar worker?

We first define the following events:

- $F$: randomly selected worker who is female
- $W$: randomly selected white-collar worker
We are trying to find the probability of randomly selecting a female worker who is also a white-collar worker. This can be expressed as \( P(F \cap W) \).

According to the given data, we have:

\[
P(F) = 10.3\% = 0.103 \\
P(W|F) = 30.7\% = 0.307
\]

Now, using the Multiplicative Rule of Probability, we get:

\[
P(F \cap W) = P(F)P(W|F) = (0.103)(0.307) = 0.0316 = 3.16\%
\]

Thus, 3.16% of all employed workers were white-collar female workers.

Suppose a coin was tossed twice, and the observed face was recorded on each toss. The following events are defined:

\[A : \text{first toss is a head}\]
\[B : \text{second toss is a head}\]

2. Does knowing that event \( A \) has occurred affect the probability of the occurrence of \( B \)?

The sample space of this experiment is \( S = \{HH, HT, TH, TT\} \), and each of these simple events has a probability of 0.25. So far we know the following information:

\[
P(A) = P(HT) + P(HH) = \frac{1}{4} + \frac{1}{4} = 0.5 \\
P(B) = P(TH) + P(HH) = \frac{1}{4} + \frac{1}{4} = 0.5 \\
A \cap B = \{HH\} \\
P(A \cap B) = 0.25
\]

Now, what is the conditional probability? It is as follows:

\[
P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}
\]

What does this tell us? It tells us that \( P(B) = \frac{1}{2} \) and also that \( P(B|A) = \frac{1}{2} \). This means knowing that the first toss resulted in heads does not affect the probability of the second toss being heads. In other words, \( P(B|A) = P(B) \).

When this occurs, we say that events \( A \) and \( B \) are independent events.
Independence

If event $B$ is independent of event $A$, then the occurrence of event $A$ does not affect the probability of the occurrence of event $B$. Therefore, we can write $P(B) = P(B|A)$.

Recall that $P(B|A) = \frac{P(B \cap A)}{P(A)}$. Therefore, if $B$ and $A$ are independent, the following must be true:

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = P(B)$$

$$P(A \cap B) = P(A) \cdot P(B)$$

That is, if two events are independent, $P(A \cap B) = P(A) \cdot P(B)$.

Determining Independence

The table below gives the number of physicists (in thousands) in the US cross-classified by specialty ($P_1, P_2, P_3, P_4$) and base of practice ($B_1, B_2, B_3$). (Remark: The numbers are absolutely hypothetical and do not reflect the actual numbers in the three bases.) Suppose a physicist is selected at random. Is the event that the physicist selected is based in academia independent of the event that the physicist selected is a nuclear physicist? In other words, is event $B_1$ independent of event $P_3$?

**Table 1.1:**

<table>
<thead>
<tr>
<th>Specialty</th>
<th>Academia ($B_1$)</th>
<th>Industry ($B_2$)</th>
<th>Government ($B_3$)</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>General Physics</td>
<td>10.3</td>
<td>72.3</td>
<td>11.2</td>
<td>93.8</td>
</tr>
<tr>
<td>Semiconductors</td>
<td>11.4</td>
<td>0.82</td>
<td>5.2</td>
<td>17.42</td>
</tr>
<tr>
<td>Nuclear Physics</td>
<td>1.25</td>
<td>0.32</td>
<td>34.3</td>
<td>35.87</td>
</tr>
<tr>
<td>Astrophysics ($P_4$)</td>
<td>0.42</td>
<td>31.1</td>
<td>35.2</td>
<td>66.72</td>
</tr>
<tr>
<td>Total</td>
<td>23.37</td>
<td>104.54</td>
<td>85.9</td>
<td>213.81</td>
</tr>
</tbody>
</table>

**Figure:** A table showing the number of physicists in each specialty (thousands). These data are hypothetical.

We need to calculate $P(B_1|P_3)$ and $P(B_1)$. If these two probabilities are equal, then the two events $B_1$ and $P_3$ are indeed independent. From the table, we find the following:

$$P(B_1) = \frac{23.37}{213.81} = 0.109$$
and

\[ P(B_1|P_3) = \frac{P(B_1 \cap P_3)}{P(P_3)} = \frac{1.25}{35.87} = 0.035 \]

Thus, \( P(B_1|P_3) \neq P(B_1) \), and so events \( B_1 \) and \( P_3 \) are not independent.

**Caution!** If two outcomes of one event are mutually exclusive (they have no overlap), they are not independent. If you know that outcomes \( A \) and \( B \) do not overlap, then knowing that \( B \) has occurred gives you information about \( A \) (specifically that \( A \) has not occurred, since there is no overlap between the two events). Therefore, \( P(A|B) \neq P(A) \).

**Example**

A college class has 42 students of which 17 are male and 25 are female. Suppose the teacher selects two students at random from the class. Assume that the first student who is selected is not returned to the class population.

**Example1**

What is the probability that the first student selected is female and the second is male?

Here we can define two events:

- \( F_1 \) : first student selected is female
- \( M_2 \) : second student selected is male

In this problem, we have a conditional probability situation. We want to determine the probability that the first student selected is female and the second student selected is male. To do so, we apply the Multiplicative Rule:

\[ P(F_1 \cap M_2) = P(F_1)P(M_2|F_1) \]

Before we use this formula, we need to calculate the probability of randomly selecting a female student from the population. This can be done as follows:

\[ P(F_1) = \frac{25}{42} = 0.595 \]

Now, given that the first student selected is not returned back to the population, the remaining number of students is 41, of which 24 are female and 17 are male.

Thus, the conditional probability that a male student is selected, given that the first student selected was a female, can be calculated as shown below:

\[ P(M_2|F_1) = \frac{17}{41} = 0.415 \]
Substituting these values into our equation, we get:

\[
P(F_1 \cap M_2) = P(F_1)P(M_2|F_1) \\
= (0.595)(0.415) \\
= 0.247 \\
= 24.7\%\
\]

We conclude that there is a probability of 24.7% that the first student selected is female and the second student selected is male.

**Review**

For 1-4, you toss a coin and roll a die. Find each of the following probabilities:

1. \( P(\text{a head and a 4}) \)
2. \( P(\text{a head and an odd number}) \)
3. \( P(\text{a tail and a number larger than 1}) \)
4. \( P(\text{a tail and a number less than 3}) \)

5. Two fair dice are tossed, and the following events are identified:
   
   \( A : \text{sum of the numbers is odd} \)
   
   \( B : \text{sum of the numbers is 9, 11, or 12} \)

   a. Are events \( A \) and \( B \) independent? Why or why not?
   b. Are events \( A \) and \( B \) mutually exclusive? Why or why not?

6. The probability that a certain brand of television fails when first used is 0.1. If it does not fail immediately, the probability that it will work properly for 1 year is 0.99. What is the probability that a new television of the same brand will last 1 year?

7. A coin is tossed 3 times. Determine the probability of getting the following results:
   
   a. head, head, head
   b. Head, tail, head

8. Given that a couple decides to have 4 children, none of them adopted. What is the probability their children will be born in the order boy, girl, boy, girl?

9. Two archers, John and Mary, shoot at a target at the same time. John hits the bulls-eye 70% of the time and Mary hits the bulls-eye 90% of the time. Find the probability that
   
   a. They both hit the bulls-eye
   b. They both miss the bulls-eye
   c. John hits the bulls-eye but Mary misses
   d. Mary hits the bulls-eye but John misses

10. A box contains 8 red and 4 blue balls. Two balls are randomly selected from the box without replacement. Determine each of the following probabilities:
   
   a. Both are red
   b. The first is blue and the second is red
   c. A blue and a red are obtained

11. A hat contains tickets with numbers 1, 2, 3, …… 20 printed on them. If three tickets are drawn from the hat without replacement, determine the probability that none of them are primes.

12. Suppose you have a spinner with 4 sections: Black, black, yellow and red. You spin the spinner twice;
a. What is the probability that black appears on both spins?
b. What is the probability that red appears on both spins?
c. What is the probability that different colors appear on both spins?
d. What is the probability that black appears on either spin?

13. Bag A contains 4 red and 3 blue tickets. Bag B contains 3 red and 1 blue ticket. A bag is randomly selected by tossing a coin and one ticket is removed from it. Using a tree diagram, determine the probability that the ticket chosen is blue.

14. Matthew and Chris go out for dinner. They roll a die and if the number of dots comes up even Matthew will pay and if the number of dots comes up odd Chris will pay. They roll the die twice, once for the decision about who pays for dinner and the second roll for the decision about who leaves the tip. A possible outcome lists who pays for dinner and then who leaves the tip. For example a possible outcome could be Chris, Chris.
   a. List all the possible simple events in this sample space.
   b. Are these events equally likely?
   c. What is the probability that Matthew will have to pay for lunch and leave the tip?

15. When a fair die is tossed each of the six sides is equally likely to land face up. Suppose you toss two die, one red and the other blue. Explain if the following pairs of events are disjoint.
   a. A = red die is 4 and B = blue die is 3
   b. A = red die and blue die sum to 5 and B = blue die is 1
   c. A = red die and blue die sum to 5 and B = red die is 5

16. Amy is taking a statistics class and a biology class. Suppose her probabilities of getting A’s are: P(grade of A in statistics) = .65 P(grade of A in biology) = .70 P(grade of A in statistics and a grade of A in biology) = .50
   a. Are the events “a grade of A in statistics” and a grade of A in biology independent? Explain.
   b. Find the probability that Amy will get at least one A between her statistics and biology classes.

Review (Answers)

To view the Review answers, open this PDF file and look for section 3.5.