Calculus Concept Collection - Chapter 9

Sequences, Their Limits and Convergence

Answers

1. $9, 11, 13, 15, 17;\ a_{10} = 27.$
2. $\frac{1}{2}, \frac{1}{4}, \frac{1}{8};\ a_8 = \frac{1}{256}.$
3. $1, 3, 6, 10;\ a_{20} = 210.$
4. $a_n = (-1)^n 2$
5. $a_n = n(n+5)$
6. $a_n = \frac{n(n+3)}{2}$
7. convergent; Limit is 0
8. convergent; Limit is 6
9. No limit exists.
10. divergent
11. convergent; Limit is 0
12. No limit exists.
13. No limit exists.
14. convergent; Limit is 0
15. By definition of absolute value, $-|a_n| \le a_n \le |a_n|$. Then take limits of all three terms:

$$\lim_{n\to+\infty} (-1|a_n|) \le \lim_{n\to+\infty} a_n \le \lim_{n\to+\infty} |a_n|$$

$$-\lim_{n\to+\infty} (|a_n|) \le \lim_{n\to+\infty} a_n \le \lim_{n\to+\infty} |a_n|$$

$$0 \le \lim_{n\to+\infty} a_n \le 0$$

Sequences: Limit Tools for Determining Convergence

Answers

1. converges, $\lim\limits_{n\to\infty} \sqrt[8n]{3n} = 1$: $\lim\limits_{n\to\infty} \sqrt[8n]{3n} = \lim\limits_{n\to\infty} e^{\ln(\sqrt[8n]{3n})} = \lim\limits_{n\to\infty} e^{\frac{1}{8n}\ln(3n)} = e^0 = 1$

   (Use L'Hopital's rule: $\lim\limits_{n\to\infty} \dfrac{\ln(3n)}{8n} = \lim\limits_{n\to\infty} \dfrac{3}{3n\square 8n} = 0$).

2. converges, $\lim\limits_{n\to\infty} \dfrac{3n^2-5}{n^2-5n+2} = \lim\limits_{n\to\infty} \dfrac{6n}{2n-5} = 3$ (Using L'Hopital's rule)

3. converges, $\lim\limits_{n\to\infty} \dfrac{n}{e^n} = \lim\limits_{n\to\infty} \dfrac{1}{e^n} = 0$ (Using L'Hopital's rule)

4. converges, $\lim\limits_{n\to\infty} \dfrac{\ln(n)}{n^2} = \lim\limits_{n\to\infty} \dfrac{1}{n^3} = 0$ (Using L'Hopital's rule)

5. converges, $\lim\limits_{n\to\infty} \left(1-\dfrac{b}{n}\right)^{an} = e^{ab}$: $\lim\limits_{n\to\infty} \left(1-\dfrac{b}{n}\right)^{an} = \lim\limits_{n\to\infty} e^{\ln\left(1-\frac{b}{n}\right)^{an}} = e^{\lim\limits_{n\to\infty} \frac{a\ln\left(1-\frac{b}{n}\right)}{1/n}} = e^{ab}$

   (Use L'Hopital's rule: $\lim\limits_{n\to\infty} \dfrac{a\ln\left(1-\frac{b}{n}\right)}{1/n} = \lim\limits_{n\to\infty} \dfrac{\frac{a}{1-b/n}\square\frac{b}{n^2}}{-1/n^2} = \lim\limits_{n\to\infty} \dfrac{ab}{1-\frac{b}{n}} = ab$)

6. converges, $\lim\limits_{n\to\infty} \dfrac{n}{n^2+1} = 0$: Use L'Hopital Rule or Squeeze Theorem,

   $0 < \lim\limits_{n\to\infty} \dfrac{n}{n^2+1} < \lim\limits_{n\to\infty} \dfrac{n}{n^2} = 0$

7. converges, $\lim\limits_{n\to\infty} \dfrac{\cos n}{n} = 0$: Use Squeeze Theorem

   $\lim\limits_{n\to\infty} \left(-\dfrac{1}{n}\right) \le \lim\limits_{n\to\infty} \dfrac{\cos n}{n} \le \lim\limits_{n\to\infty} \left(\dfrac{1}{n}\right) \Rightarrow 0 \le \lim\limits_{n\to\infty} \dfrac{\cos n}{n} \le 0$

8. converges, $\lim\limits_{n\to\infty} \dfrac{\sin n}{n} = 0$: Use Squeeze Theorem

   $\lim\limits_{n\to\infty} \left(-\dfrac{1}{n}\right) \le \lim\limits_{n\to\infty} \dfrac{\sin n}{n} \le \lim\limits_{n\to\infty} \left(\dfrac{1}{n}\right) \Rightarrow 0 \le \lim\limits_{n\to\infty} \dfrac{\sin n}{n} \le 0$

9. converges, $\lim\limits_{n\to\infty} \dfrac{7}{n!} = 0$: $0 < \lim\limits_{n\to\infty} \dfrac{7}{n!} < \lim\limits_{n\to\infty} \dfrac{7}{n} \Rightarrow 0 < \lim\limits_{n\to\infty} \dfrac{7}{n!} < 0$

10. diverges, $\lim\limits_{n\to\infty} \left(\dfrac{n^2}{n+1} - \dfrac{1}{n+1}\right)$: $\lim\limits_{n\to\infty} \left(\dfrac{n^2}{n+1} - \dfrac{1}{n+1}\right) = \lim\limits_{n\to\infty} \left(\dfrac{n^2}{n+1}\right) - \lim\limits_{n\to\infty} \left(\dfrac{1}{n+1}\right) = \infty - 0 = \infty$

11. diverges, $\lim\limits_{n\to\infty}\dfrac{\ln(n+7)}{n^{1/n}}$: $\lim\limits_{n\to\infty}\dfrac{\ln(n+7)}{n^{1/n}}=\dfrac{\lim\limits_{n\to\infty}\ln(n+7)}{\lim\limits_{n\to\infty}n^{1/n}}=\dfrac{\infty}{1}=\infty$.

$$y=n^{1/n}\Rightarrow \ln y=\frac{1}{n}\ln n\Rightarrow y=e^{\frac{1}{n}\ln n}\Rightarrow \lim_{n\to\infty}y=\lim_{n\to\infty}e^{\frac{1}{n}\ln n}=\lim_{n\to\infty}e^{-\frac{1}{n^2}}=1.$$

12. diverges, $\lim\limits_{n\to\infty}\dfrac{n^3+n^2-n+1)}{n^2+1}=\lim\limits_{n\to\infty}\left(n+1-\dfrac{2n)}{n^2+1}\right)=\infty$

13. converges, $\lim\limits_{n\to\infty}\dfrac{\sin^2 n}{3^n}=0$: Using the Squeeze Theorem $0\le\lim\limits_{n\to\infty}\dfrac{\sin^2 n}{3^n}\le\lim\limits_{n\to\infty}\dfrac{1}{3^n}=0$.

14. converges, $\lim\limits_{n\to\infty}\left(3-\dfrac{1}{3^n}\right)\left(2+\dfrac{1}{2^n}\right)=6$:

$$\lim_{n\to\infty}\left(3-\frac{1}{3^n}\right)\left(2+\frac{1}{2^n}\right)=\lim_{n\to\infty}\left(3-\frac{1}{3^n}\right)\cdot\lim_{n\to\infty}\left(2+\frac{1}{2^n}\right)=3\cdot 2=6$$

15. converges, $\lim\limits_{n\to+\infty}\left(\dfrac{11}{n}-\dfrac{8}{n^2}\right)=0$:

$$\lim_{n\to+\infty}\left(\frac{11}{n}-\frac{8}{n^2}\right)=11\cdot\lim_{n\to+\infty}\left(\frac{1}{n}\right)-8\cdot\lim_{n\to+\infty}\left(\frac{1}{n^2}\right)=11\cdot 0-8\cdot 0=0.$$

Introduction to Infinite Series

Answers

1. $\frac{3}{9}=0.\overline{3}=\sum_{n=1}^{\infty}\frac{3}{10}\left(\frac{1}{10}\right)^{n-1}=\sum_{n=1}^{\infty}3\left(\frac{1}{10}\right)^{n}$

2. $\frac{1}{11}=0.\overline{09}=\sum_{k=1}^{\infty}\frac{9}{100}\left(\frac{1}{100}\right)^{k-1}$

3. $0.\overline{037}=\sum_{n=1}^{\infty}0.037\left(\frac{1}{1000}\right)^{n-1}=\sum_{n=1}^{\infty}37\left(\frac{1}{1000}\right)^{n}$

4. $0.\overline{1573}=\sum_{n=1}^{\infty}\frac{1573}{10000}10^{-4(n-1)}=\sum_{n=1}^{\infty}1.573\cdot 10^{-4n+3}$

5. $0.12012001200012....=0.12\cdot 10^{0}+0.12\cdot 10^{-3}+0.12\cdot 10^{-7}+0.12\cdot 10^{-12}...=\sum_{n=1}^{\infty}0.12\cdot 10^{-\frac{(n-1)(n+4)}{2}}$

6. $-7-3+1+5+9+...=\sum_{n=1}^{\infty}\left[-7+4(n-1)\right]$; positive/negative term arithmetic series with $t_1=-7,\ d=4$.

7. $\sum_{k=1}^{\infty}\left(\frac{3}{5}\right)^{k-1}$; positive term geometric series with $a=1,\ r=\frac{3}{5}$.

8. $\sum_{k=1}^{\infty}\frac{k^3}{k^3-5}$; pos/neg term series (1 negative term).

9. $\sum_{k=1}^{\infty}\frac{4^{k+2}}{9^{k-1}}=\sum_{k=1}^{\infty}4^3\frac{4^{k-1}}{9^{k-1}}=\sum_{k=1}^{\infty}64\left(\frac{4}{9}\right)^{k-1}$; positive term geometric series with $a=64,\ r=\frac{4}{9}$.

10. $\sum_{k=1}^{\infty}\frac{k+2}{5^{k-1}}$; positive term series.

    $S_1=3,\ S_2=\frac{19}{5},\ S_3=4$

11. $\sum_{n=1}^{\infty}5\left(-\frac{1}{2}\right)^{n-1}$; pos/neg term (alternating) geometric series with $a=5,\ r=-\frac{1}{2}$.

    $S_n=a\frac{1-r^n}{1-r}$: $S_1=5,\ S_2=\frac{5}{2},\ S_3=\frac{15}{4}$.

12. $\sum_{n=1}^{\infty}\left(-13+3(n-1)\right)$; positive/negative term arithmetic series with $t_1=-13,\ d=3$.

$S_n=\dfrac{n\left[2t_1+d(n-1)\right]}{2}$: $S_1=-13,\ S_2=-23,\ S_3=-30$.

13. $\sum_{i=1}^{\infty}i^3$; positive term series where $\sum_{i=1}^{n}i^3=\left[\dfrac{n(n+1)}{2}\right]^2=\left[\sum_{i=1}^{n}i\right]^2$.

$\sum_{i=1}^{n}i^3=\left[\dfrac{n(n+1)}{2}\right]^2$: $S_1=1,\ S_2=9,\ S_3=36$

14. $\sum_{n=1}^{\infty}\dfrac{7}{n}$; positive term harmonic series.

$S_1=7,\ S_2=\dfrac{21}{2},\ S_3=\dfrac{77}{6}$

15. $\sum_{n=1}^{\infty}\dfrac{30}{n^2}$; positive term p-series with $p=2$.

$S_1=30,\ S_2=\dfrac{75}{2},\ S_3=\dfrac{245}{6}$

Determining Convergence or Divergence of an Infinite Series

Answers

1. $3+\frac{3}{10}+\frac{3}{10^2}+\frac{3}{10^3}+\ldots$ converges because it is a geometric series with $-1<r=\frac{1}{10}<1$.

   It converges to $S=\frac{a}{1-r}=\frac{3}{1-0.1}=\frac{3}{0.9}=\frac{10}{3}$

2. $\sum_{k=1}^{\infty}\left(\frac{3}{5}\right)^{k-1}$ converges because it is a geometric series with $-1<r=\frac{3}{5}<1$.

   It converges to $S=\frac{a}{1-r}=\frac{1}{1-0.6}=\frac{1}{0.4}=2.5$

3. $\sum_{k=1}^{+\infty}\left(-\frac{2}{3}\right)^{k-1}$ converges because it is a geometric series with $-1<r=-\frac{2}{3}<1$.

   It converges to $S=\frac{a}{1-r}=\frac{1}{1+2/3}=\frac{3}{5}$

4 $\sum_{k=1}^{\infty}\frac{k^3}{k^3-5}$ diverges because the nth term divergence test yields

$\lim_{k\to\infty}\frac{k^3}{k^3-5}=\lim_{k\to\infty}\frac{1}{1-5/k^3}=1\neq 0$.

5. $\sum_{k=1}^{\infty}\frac{4^{k+2}}{9^{k-1}}=\sum_{k=1}^{\infty}4^3\left(\frac{4}{9}\right)^{k-1}$ converges because it is a geometric series with $-1<r=\frac{4}{9}<1$.

   It converges to $S=\frac{a}{1-r}=\frac{4^3}{1-4/9}=\frac{64\square 9}{5}=\frac{576}{5}=115.2$.

6. $7+\frac{7}{8}+\frac{7}{8^2}+\frac{7}{8^3}+\ldots+\frac{7}{8^{i-1}}+\ldots=\sum_{i=1}^{\infty}7\left(\frac{1}{8}\right)^{i-1}$ converges because it is a geometric series with $-1<r=\frac{1}{8}<1$.

It converges to $S=\frac{a}{1-r}=\frac{7}{1-1/8}=\frac{56}{7}$

7. $\sum_{k=1}^{+\infty}9^{k-1}$ diverges because it is a geometric series with $r=9>1$.

8. $-\frac{3}{4}+\frac{3}{4^2}-\frac{3}{4^3}+\ldots+\frac{3(-1)^k}{4^k}+\ldots=\sum_{k=1}^{\infty}-\frac{3}{4}\left(-\frac{1}{4}\right)^{k-1}$ converges because it is a geometric series with $-1<r=-\frac{1}{4}<1$.

It converges to $S=\frac{a}{1-r}=\frac{-3/4}{1+1/4}=-\frac{3}{5}$

9. $3+2+\frac{4}{3}+\frac{8}{9}+\ldots=\sum_{n=1}^{\infty}3\left(\frac{2}{3}\right)^{n-1}$ converges because it is a geometric series with $-1<r=\frac{2}{3}<1$.

It converges to $S=\frac{a}{1-r}=\frac{3}{1-2/3}=9$

10. $-2+\frac{5}{2}-\frac{25}{8}+\frac{125}{32}+\ldots=\sum_{k=1}^{\infty}-2\left(-\frac{5}{4}\right)^{k-1}$ diverges because it is a geometric series with $r=-\frac{5}{4}<-1$.

11. $\sum_{n=1}^{\infty}\frac{n}{n+7}$ diverges because the nth term divergence test yields $\lim_{n\to\infty}\frac{n}{n+7}=\lim_{n\to\infty}\frac{1}{1+5/n}=1\neq 0$.

12. $\sum_{n=1}^{\infty} \frac{n+1}{2n-3}$ diverges because the nth term divergence test yields

$$\lim_{n\to\infty} \frac{n+1}{2n-3} = \lim_{n\to\infty} \frac{1+1/n}{2-3/n} = \frac{1}{2} \neq 0.$$

13. $\sum_{n=1}^{\infty} \sqrt[n]{3} = \sum_{n=1}^{\infty} 3^{1/n}$ diverges because the nth term divergence test yields

$$\lim_{n\to\infty} 3^{1/n} = 3^0 = 1 \neq 0$$

14. $\sum_{k=1}^{\infty} 3\sin(x)^{k-1}$ has the form of a geometric series with $a = 3,\ r = \sin(x)$. Convergence requires that $\left| r = \sin(x) < 1 \right|$, which means $x \neq \pm(2n-1)\frac{\pi}{2},\ n = 1, 2, 3\ldots$.

For acceptable values of x, the series converges to $S = \frac{a}{1-r} = \frac{3}{1-\sin(x)}$.

15. $0.99999999\ldots = \sum_{n=1}^{\infty} 0.9\left(\frac{1}{10}\right)^{k-1}$ is a geometric which converges to

$$S = \frac{a}{1-r} = \frac{0.9}{1-0.1} = \frac{0.9}{0.9} = 1\ .$$

Some Properties of Infinite Series

Answers

1. $\sum_{n=1}^{\infty} \frac{3n}{n+4}$ diverges by the nth term divergence test: $\lim_{n\to\infty} \frac{3n}{n+4} = \lim_{n\to\infty} \frac{3}{1+4/n} = 3$.

2. $\sum_{n=1}^{\infty} \frac{2+(-4)^n}{5^n}$ converges with sum $\frac{2}{9}$: $\sum_{n=1}^{\infty} \frac{2+(-4)^n}{5^n} = \sum_{n=1}^{\infty} \frac{2}{5}\left(\frac{2}{5}\right)^{n-1} + \sum_{n=1}^{\infty} \frac{-4}{5}\left(\frac{-4}{5}\right)^{n-1}$ two convergent geometric series.

3. $\sum_{n=1}^{\infty} \frac{n}{4}$ diverges by the nth term divergence test.

4. $\sum_{n=1}^{\infty} \frac{3n}{n^4}$ is a convergent p-series.

5. $\sum_{n=1}^{\infty} \left[\frac{n^2+2}{n^2+1} + 5\left(\frac{1}{2}\right)^{n-1}\right]$ diverges: it is the sum of a divergent series ($\lim_{n\to\infty} \frac{n^2+2}{n^2+1} = 1$) and a convergent geometric series.

6. $\sum_{n=1}^{\infty} \frac{n^2+3n+2}{4n^2+1}$ diverges: $\lim_{n\to\infty} \frac{n^2+3n+2}{4n^2+1} = \frac{1}{4}$.

7. $\sum_{n=1}^{\infty} \left[\frac{2}{n^2} + 7\left(\frac{1}{3}\right)^{n+3}\right]$ converges with sum $\frac{\pi^2}{3} + \frac{21}{2}\left(\frac{1}{3}\right)^4$: sum of a convergent p-series and a convergent geometric series. $\sum_{n=1}^{\infty} \left[\frac{2}{n^2}\right] = 2\square\frac{\pi^2}{6} = \frac{\pi^2}{3}$ and

$$\sum_{n=1}^{\infty} \left[7\left(\frac{1}{3}\right)^{n+3}\right] = \sum_{n=1}^{\infty} \left[7\left(\frac{1}{3}\right)^4 \left(\frac{1}{3}\right)^{n-1}\right] = \frac{21}{2}\left(\frac{1}{3}\right)^4.$$

8. $\sum_{k=2}^{\infty}\left(\left(-\frac{2}{3}\right)^{k-1}+\frac{1}{5^{k-1}}\right)$ converges with sum $-\frac{3}{20}$: sum of two convergent geometric series. $\sum_{k=2}^{\infty}\left(\left(-\frac{2}{3}\right)^{k-1}+\frac{1}{5^{k-1}}\right)=\sum_{k=1}^{\infty}\left(-\frac{2}{3}\left(-\frac{2}{3}\right)^{k-1}+\frac{1}{5}\left(\frac{1}{5}\right)^{k-1}\right)=-\frac{2}{5}+\frac{1}{4}=-\frac{3}{20}$.

9. $\sum_{k=1}^{\infty}\left(\frac{4}{5^{k-1}}-\frac{2^k}{3^k}\right)$ converges with sum 3: sum of two convergent geometric series.

10. $\sum_{n=3}^{\infty}\frac{4}{n^2}=\frac{2\pi^2}{3}-5:\sum_{n=3}^{\infty}\frac{4}{n^2}=\sum_{n=1}^{\infty}\frac{4}{n^2}-(4+1)=4\sum_{n=1}^{\infty}\frac{1}{n^2}-(4+1)=4\frac{\pi^2}{6}-5=\frac{2\pi^2}{3}-5$.

11. $\sum_{n=1}^{\infty}\frac{3n}{n+4}=\sum_{k=7}^{\infty}\frac{3(k+1-7)}{k+(1-7)+4}=\sum_{k=7}^{\infty}\frac{3(k-6)}{(k-6)+4}=\sum_{k=7}^{\infty}\frac{3k-18}{k-2}$.

12. $\sum_{n=1}^{\infty}\frac{2^{n-1}}{n!}=\sum_{n=0}^{\infty}\frac{2^n}{(n+1)!}$.

13. a. the even summands of $\sum_{n=1}^{\infty}\frac{1}{n}$: $\sum_{n=1}^{\infty}\frac{1}{2n}$

b. the odd summands of $\sum_{n=1}^{\infty}\frac{1}{n}$: $\sum_{n=1}^{\infty}\frac{1}{2n+1}$

14. $\sum_{n=1}^{\infty}\left[\frac{2}{n^2}+7\left(\frac{1}{3}\right)^{n+3}\right]=\sum_{k=10}^{\infty}\left[\frac{2}{(k+1-10)^2}+7\left(\frac{1}{3}\right)^{k+(1-10)+3}\right]=\sum_{k=10}^{\infty}\left[\frac{2}{(k-9)^2}+7\left(\frac{1}{3}\right)^{k-6}\right]$

15. $\sum_{n=5}^{\infty}\frac{n^2+3n+2}{4n^2+1}=\sum_{k=1}^{\infty}\frac{(k+5-1)^2+3(k+5-1)+2}{4(k+5-1)^2+1}=\sum_{k=1}^{\infty}\frac{(k+4)^2+3(k+4)+2}{4(k+4)^2+1}$.

Positive term series: the Integral Test

Answers

1. Maria is correct that the series converges. She made an error by saying that the value of the related integral gives the sum of the infinite series. However, $\sum_{k=1}^{\infty}\frac{3}{(k^2)}=3+\frac{3}{4}+\frac{3}{9}+\ldots$ is greater than 3: $\sum_{k=1}^{\infty}\frac{3}{(k^2)}=3\sum_{k=1}^{\infty}\frac{1}{(k^2)}=3\cdot\frac{\pi^2}{6}=\frac{\pi^2}{2}$.

2. $\sum_{n=3}^{\infty}\frac{5}{n-2}$ diverges: $f(x)=\frac{5}{x-2}$ is continuous, positive, and decreasing ($f'(x)=-\frac{5}{(x-2)^2}<0,\ x\geq 3$):

   $$\int_3^{\infty}\frac{5}{x-2}dx=\lim_{p\to\infty}\int_3^{p}\frac{5}{x-2}dx=5\cdot\lim_{p\to\infty}\left[\ln(x-2)\right]_3^p=5\cdot\lim_{p\to\infty}\left[\ln(p-2)-0\right]=\infty$$

3. $\sum_{n=1}^{\infty}\frac{n+2}{n+1}$ diverges: $f(x)=\frac{x+2}{x+1}$ is continuous, positive, and decreasing ($f'(x)=-\frac{1}{(x+1)^2}<0,\ x\geq 1$):

   $$\int_1^{\infty}\frac{x+2}{x+1}dx=\lim_{p\to\infty}\int_1^{p}\frac{x+2}{x+1}dx=\lim_{p\to\infty}\left[x+1+\ln(x+1)\right]_1^p=\lim_{p\to\infty}\left[p-1+\ln(\frac{p+1}{2})\right]=\infty$$

4. $\sum_{n=1}^{\infty}\frac{n}{3n^2+2}$ diverges: $f(x)=\frac{x}{3x^2+2}$ is continuous, positive, and decreasing.

   $$\int_1^{\infty}\frac{x}{3x^2+2}dx=\lim_{p\to\infty}\int_1^{p}\frac{x}{3x^2+2}dx=\frac{1}{6}\cdot\lim_{p\to\infty}\left[\ln(3x^2+2)\right]_1^p=\frac{1}{6}\cdot\lim_{p\to\infty}\left[\ln(3p^2+2)-\ln 5\right]=\infty$$

5. $\frac{1}{3}\ln 3+\frac{1}{4}\ln 4+\frac{1}{5}\ln 5+\ldots=\sum_{n=1}^{\infty}\frac{\ln(n+2)}{n+2}$ diverges: $f(x)=\frac{\ln(x+2)}{x+2}$ is continuous, positive, and decreasing $f'(x)=-\frac{1-\ln(x+2)}{(x+2)^2}<0,\ x\geq 1$.

   $$\int_1^{\infty}\frac{\ln(x+2)}{x+2}dx=\lim_{p\to\infty}\int_1^{p}\frac{\ln(x+2)}{x+2}dx=\lim_{p\to\infty}\left[\frac{(\ln(x+2))^2}{2}\right]_1^p=\lim_{p\to\infty}\left[\frac{(\ln(p+2))^2}{2}-\frac{(\ln 3)^2}{2}\right]=\infty$$

6. $\sum_{n=1}^{\infty}\frac{1}{(n+1)\ln(n+1)}$ diverges: $f(x)=\frac{1}{(x+1)\ln(x+1)}$ is continuous, positive, and decreasing.

$$\int_1^\infty f(x)dx = \lim_{p\to\infty}\int_1^p \frac{1}{(x+1)\ln(x+1)}dx = \lim_{p\to\infty}\left[\ln(\ln(x+1))\right]_1^p = \lim_{p\to\infty}\left[\ln(\ln(p+1)) - \ln(\ln 2)\right] = \infty$$

7. $\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^2}$ converges: $f(x) = \frac{1}{x(\ln x)^2}$, $x \ge 2$ is continuous, positive, and decreasing.

$$\int_2^\infty f(x)dx = \lim_{p\to\infty}\int_2^p \frac{1}{x(\ln x)^2}dx = \lim_{p\to\infty}\left[-\frac{1}{\ln p} + \frac{1}{\ln 2}\right] = \frac{1}{\ln 2}$$

8. $\sum_{n=1}^{\infty}\frac{1}{n^{1/4}}$ diverges: $f(x) = \frac{1}{x^{1/4}}$, $x \ge 1$ is continuous, positive, and decreasing ($f'(x) = -\frac{1}{4x^{5/4}} < 0$).

$$\int_1^\infty f(x)dx = \lim_{p\to\infty}\int_1^p \frac{1}{x^{1/4}}dx = \lim_{p\to\infty}\left[\frac{4}{3}p^{3/4} - \frac{4}{3}\right] = \infty.$$

9. $\sum_{k=1}^{\infty}\frac{7}{\sqrt[5]{k^2}}$ diverges. $f(x) = \frac{7}{\sqrt[5]{x^2}} = \frac{7}{x^{2/5}}$, $x \ge 1$ is continuous, positive, and decreasing ($f'(x) = -\frac{14}{5x^{7/5}} < 0,\ x \ge 1$).

$$\int_1^\infty f(x)dx = \lim_{p\to\infty}\int_1^p \frac{7}{x^{2/5}}dx = \lim_{p\to\infty}\left[\frac{35}{3}p^{3/5} + \frac{35}{3}\right] = \infty.$$

10. $\sum_{n=1}^{\infty} e^{-n}$ converges: $f(x) = e^{-x}$, $x \ge 1$ is continuous, positive, and decreasing ($f'(x) = -e^{-x} < 0,\ x \ge 1$).

$$\int_1^\infty f(x)dx = \lim_{p\to\infty}\int_1^p e^{-x}dx = \lim_{p\to\infty}\left[-e^{-p} + e^{-1}\right] = e^{-1}.$$

11. $\sum_{k=1}^{\infty}\frac{1}{(3k-1)^{5/2}}$ converges: $f(x) = \frac{1}{(3x-1)^{5/2}}$, $x \ge 1$ is continuous, positive, and decreasing ($f'(x) = -\frac{15}{2(3x-1)^{7/2}} < 0,\ x \ge 1$).

$$\int_1^\infty f(x)dx = \lim_{p\to\infty}\int_1^p \frac{1}{(3x-1)^{5/2}}dx = \lim_{p\to\infty}\left[-\frac{2}{3(3p-1)^{3/2}} + \frac{2}{3(2)^{3/2}}\right] = \frac{\sqrt{2}}{6}.$$

12. $\sum_{n=2}^{\infty}\frac{\ln n}{n^2}$ diverges: $f(x)=\frac{\ln x}{x^2},\ x\ge 2$ is continuous, positive, and decreasing ($f'(x)=-\frac{x(2\ln x-1)}{x^4}<0,\ x\ge 2$).

$$\int_1^{\infty} f(x)dx=\lim_{p\to\infty}\int_2^{p}\frac{\ln x}{x^2}dx=\lim_{p\to\infty}\left[\frac{x^3}{3}(\ln x-\frac{1}{3})\right]_2^{p}=\infty\ .$$

13. $\sum_{n=1}^{\infty}\frac{1}{n^2+4}$ converges: $f(x)=\frac{1}{x^2+4}$ is continuous, positive, and decreasing.

$\int_1^{\infty}\frac{1}{x^2+4}dx=\lim_{p\to\infty}\int_1^{p}\frac{1}{x^2+4}dx$, use $x=2\tan\theta,\ dx=2\sec^2\theta d\theta\Rightarrow$

$$\lim_{p\to\infty}\int_1^{p}\frac{1}{x^2+4}dx=\lim_{p\to\infty}\left[\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\right]_1^{p}=\frac{1}{2}\lim_{p\to\infty}\left[\tan^{-1}\left(\frac{p}{2}\right)-\tan^{-1}\left(\frac{1}{2}\right)\right]_1^{p}=\frac{1}{2}\left[\frac{\pi}{2}-0.15\pi\right]$$

14. $\sum_{n=1}^{\infty}ne^{-n^2}$ converges: $f(x)=xe^{-x^2},\ x\ge 1$ is continuous, positive, and decreasing ($f'(x)=-(2x^2-1)e^{-x^2}<0,\ x\ge 1$).

$$\int_1^{\infty} f(x)dx=\lim_{p\to\infty}\int_1^{p}xe^{-x^2}dx=\lim_{p\to\infty}\left[-\frac{e^{-p^2}}{2}+\frac{e^{-1}}{2}\right]=\frac{e^{-1}}{2}\ .$$

15. $\sum_{n=1}^{\infty}\frac{3n+2}{n(n+1)}$ diverges: $f(x)=\frac{3x+2}{x(x+1)}$ is continuous, positive, and decreasing.

$$\int_1^{\infty}\frac{3x+2}{x(x+1)}dx=\lim_{p\to\infty}\int_1^{p}\frac{3x+2}{x(x+1)}dx=\lim_{p\to\infty}\int_1^{p}\left[\frac{2}{x}+\frac{1}{x+1}\right]dx=\lim_{p\to\infty}\left[2\ln x+\ln(x+1)\right]_1^{p}=\infty\ .$$

Positive term series: Comparison Tests

Answers

1. $\sum_{n=1}^{\infty}\frac{1}{n^3+n+3}$ converges by Comparison test using $\sum_{n=1}^{\infty}\frac{1}{n^3}$, a convergent p-series.
2. $\sum_{n=1}^{\infty}\frac{1}{2^n+3}$ converges by Comparison test using $\sum_{n=1}^{\infty}\frac{1}{2^n}=\sum_{n=1}^{\infty}\frac{1}{2}\left(\frac{1}{2}\right)^{n-1}$, a convergent geometric series.
3. $\sum_{n=1}^{\infty}\frac{3^n}{5^n+6}$ converges by Comparison test using $\sum_{n=1}^{\infty}\frac{3^n}{5^n}=\sum_{n=1}^{\infty}\frac{3}{5}\left(\frac{3}{5}\right)^{n-1}$, a convergent geometric series.
4. $\sum_{n=1}^{\infty}\frac{1}{2n^2+n+5}$ converges by Comparison test using $\sum_{n=1}^{\infty}\frac{1}{2n^2}$, a convergent p-series.
5. $\sum_{n=1}^{\infty}\frac{(\sin n)^2}{n(n+5)}$ converges by Comparison test: $\frac{(\sin n)^2}{n(n+5)}\le\frac{1}{n(n+5)}\le\frac{1}{n^2}$ a convergent p-series.
6. $\sum_{k=1}^{\infty}\frac{1}{(4k+1)^{1/2}}$ diverges by Comparison test using $\sum_{k=1}^{\infty}\frac{1}{(4k)^{1/2}}$, a divergent p-series.
7. $\sum_{n=1}^{\infty}\frac{\arctan n}{2n^3}$ converges by Comparison test: $\frac{\arctan n}{2n^3}\le\frac{\pi/2}{2n^3}=\frac{\pi}{4}\frac{1}{n^3}$, a convergent p-series.
8. $\sum_{n=1}^{\infty}\frac{\sqrt{n+4}}{n\sqrt{n+2}}$ diverges by Comparison test: $\frac{\sqrt{n+4}}{n\sqrt{n+2}}>\frac{\sqrt{n+2}}{n\sqrt{n+2}}=\frac{1}{n}$, a divergent Harmonic series.
9. $\sum_{k=1}^{\infty}\frac{2}{5k^5-4}$ converges by the Limit Comparison Test: $\lim_{n\to\infty}\frac{2/(5k^5-4)}{2/5k^5}=1$ and using $\sum_{k=1}^{\infty}\frac{2}{5k^5}$ is a convergent p-series.
10. $\sum_{k=1}^{\infty}\frac{5}{(k+1)(k+3)}=\sum_{k=1}^{\infty}\frac{5}{k^2+4k+3}$ converges by the Limit Comparison Test: $\lim_{n\to\infty}\frac{5/(k^2+4k+3)}{5/k^2}=1$ and using $\sum_{k=1}^{\infty}\frac{5}{k^2}$, which is a convergent p-series.

11. $\sum_{k=1}^{\infty}\frac{k^3+4k^2+1}{3k^6+2k^4}$ converges by the Limit Comparison Test: $\lim_{n\to\infty}\frac{\frac{k^3+4k^2+1}{3k^6+2k^4}}{1/3k^3}=\frac{1}{3}$ and using $\sum_{k=1}^{\infty}\frac{1}{k^3}$, which is a convergent p-series.

12. $\sum_{k=1}^{\infty}\frac{4k^2+3k+9}{7k^3+11}$ diverges by the Limit Comparison Test: $\lim_{n\to\infty}\frac{\frac{4k^2+3k+9}{7k^3+11}}{4k^2/7k^3}=1$ and using $\sum_{k=1}^{\infty}\frac{4k^2}{7k^3}=\sum_{k=1}^{\infty}\frac{4}{7k}$, which is a divergent Harmonic series.

13. $\sum_{n=1}^{\infty}\frac{\sqrt{n}}{2n^3+4}$ converges by the Limit Comparison Test: $\lim_{n\to\infty}\frac{\frac{\sqrt{n}}{2n^3+4}}{1/n^{5/2}}=\frac{1}{2}$ and using $\sum_{k=1}^{\infty}\frac{1}{n^{5/2}}$, which is a convergent p-series.

14. $\sum_{n=1}^{\infty}\frac{4^n+5}{7^n+13}$ converges by the Limit Comparison Test: $\lim_{n\to\infty}\frac{\frac{4^n+5}{7^n+13}}{4^n/7^n}=1$ and using $\sum_{n=1}^{\infty}\frac{4^n}{7^n}=\sum_{n=1}^{\infty}\frac{4}{7}\left(\frac{4}{7}\right)^{n-1}$, which is a convergent geometric series.

15. $\sum_{n=1}^{\infty}\frac{1}{an+b}$ with $a>0$ diverges by the Limit Comparison Test: $\lim_{n\to\infty}\frac{\frac{1}{an+b}}{1/n}=\frac{1}{a}>0$ and using $\sum_{n=1}^{\infty}\frac{1}{n}$, which is a divergent Harmonic series

PT Series: Ratio and Root Tests for Convergence

Answers

1. $\sum_{n=1}^{\infty} \frac{n^{n-1}}{(n-1)!}$ diverges by Comparison Test: $\lim_{n\to\infty} \frac{\frac{n^n}{(n)!}}{\frac{n^{n-1}}{(n-1)!}} = \frac{n}{n} = 1$, Ratio Test is inconclusive

2. $\sum_{n=1}^{\infty} \frac{n}{5^n}$ converges by the Ratio Test: $\lim_{n\to\infty} \frac{\frac{n+1}{5^{n+1}}}{\frac{n}{5^n}} = \frac{1}{5} \lim_{n\to\infty} \frac{n+1}{n} = \frac{1}{5}$.

3. $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ converges by the Ratio Test: $\lim_{n\to\infty} \frac{\frac{(n+1)^2}{2^{n+1}}}{\frac{n^2}{2^n}} = \frac{1}{2} \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^2 = \frac{1}{2}$.

4. $\sum_{n=1}^{\infty} \frac{n!}{e^n}$ diverges by the Ratio Test: $\lim_{n\to\infty} \frac{\frac{(n+1)!}{e^{n+1}}}{\frac{n!}{e^n}} = \frac{1}{e} \lim_{n\to\infty} (n+1) = \infty$.

5. $\sum_{n=1}^{\infty} \frac{n!}{1\cdot 3\cdot 5\cdots(2n-1)}$ converges by the Ratio Test: $\lim_{n\to\infty} \frac{\frac{(n+1)!}{1\cdot 3\cdot 5\cdots(2n+1)}}{\frac{n!}{1\cdot 3\cdot 5\cdots(2n-1)}} = \lim_{n\to\infty} \left(\frac{n+1}{2n+2}\right) = \frac{1}{2}$.

6. $\sum_{n=1}^{\infty} \frac{n}{3^n}$ converges by the Ratio Test: $\lim_{n\to\infty} \frac{\frac{n+1}{3^{n+1}}}{\frac{n}{3^n}} = \frac{1}{3} \lim_{n\to\infty} \left(\frac{n+1}{n}\right) = \frac{1}{3}$.

7. $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ converges by the Ratio Test:

$$\lim_{n\to\infty} \frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}} = \lim_{n\to\infty} \left(n+1 \frac{n^n}{(n+1)(n+1)^n} \right) = \lim_{n\to\infty} \left(\frac{n}{n+1} \right)^n = \lim_{n\to\infty} \left(\frac{1}{1+1/n} \right)^n = \frac{1}{e}.$$

8. $\sum_{n=1}^{\infty} \frac{n-1}{n4^n}$ converges by the Ratio Test: $\lim_{n\to\infty} \frac{\frac{n}{(n+1)4^{n+1}}}{\frac{n-1}{n4^n}} = \frac{1}{4}\lim_{n\to\infty}\left(\frac{n^2}{n^2-1}\right) = \frac{1}{4}$.

9. $\sum_{n=1}^{\infty} \frac{e^n}{n^n}$ converges by the nth Root Test: $\lim_{n\to\infty}\left(\frac{e^n}{n^n}\right)^{1/n} = \lim_{n\to\infty}\left(\frac{e}{n}\right) = 0$.

10. $\sum_{n=1}^{\infty} \frac{2^{n-3}}{4^n}$ converges by the Root Test: $\lim_{n\to\infty}\left(\frac{2^{n-3}}{4^n}\right)^{1/n} = \lim_{n\to\infty}\left(\frac{2^3}{2^3}\frac{2^{n-3}}{4^n}\right)^{1/n} = \lim_{n\to\infty}\frac{2}{4}\left(\frac{1}{2^3}\right)^{1/n} = \frac{1}{2}$.

11. $\sum_{n=1}^{\infty}\left(\frac{n^2+1}{2n^2+1}\right)^n$ converges by the Root Test: $\lim_{n\to\infty}\left(\left(\frac{n^2+1}{2n^2+1}\right)^n\right)^{1/n} = \lim_{n\to\infty}\left(\frac{n^2+1}{2n^2+1}\right) = \frac{1}{2}$.

12. $\sum_{n=1}^{\infty}\frac{3^{2n}}{7^n}$ divergent by the Root Test: $\lim_{n\to\infty}\left(\frac{3^{2n}}{7^n}\right)^{1/n} = \lim_{n\to\infty}\left(\frac{3^2}{7}\right) = \frac{9}{7} > 1$

13. $\sum_{n=1}^{\infty}\left(\frac{\ln(n^2+1)}{n}\right)^n$ converges by the Root Test:

$$\lim_{n\to\infty}\left(\left(\frac{\ln(n^2+1)}{n}\right)^n\right)^{1/n} = \lim_{n\to\infty}\left(\frac{\ln(n^2+1)}{n}\right) = \lim_{n\to\infty}\left(\frac{1}{n}\cdot\frac{2n}{n^2+1}\right) = 0.$$

14. $\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n^2}\right)^n$ converges by the Root Test: $\lim_{n\to\infty}\left(\left(\frac{1}{n}-\frac{1}{n^2}\right)^n\right)^{1/n} = \lim_{n\to\infty}\left(\frac{1}{n}-\frac{1}{n^2}\right) = 0$.

15. $\sum_{n=1}^{\infty} n\left(\frac{2}{3}\right)^n$ converges by the Root Test: $\lim_{n\to\infty}\left(n\left(\frac{2}{3}\right)^n\right)^{1/n} = \left(\frac{2}{3}\right)\lim_{n\to\infty} n^{1/n} = \frac{2}{3}$

Note: $\lim_{n\to\infty} n^{1/n} = 1$: $y = \lim_{n\to\infty} n^{1/n} \Rightarrow \ln y = \lim_{n\to\infty}\frac{\ln n}{n} = \lim_{n\to\infty}\frac{1/n}{1} = 0 \Rightarrow y = e^0 = 1$

Positive and Negative Terms: Alternating Series

Answers

1. $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$ converges by the Alternating Series Test.
2. $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2k+1}$ converges by the Alternating Series Test.
3. $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n+3}$ fails both conditions of the Alternating Series Test
4. $\sum_{k=1}^{\infty} (-\frac{5}{13})^{k-1}$ converges by the Alternating Series Test.
5. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^6}$ converges by the Alternating Series Test.
6. $\sum_{k=1}^{\infty} \frac{(-1)^k}{10^n n!}$ converges by the Alternating Series Test.
7. $\sum_{k=1}^{\infty} (-1)^k \frac{1}{\ln(n+1)}$ converges by the Alternating Series Test.
8. $\sum_{k=1}^{\infty} (-1)^k \frac{\ln(n+1)}{n+1}$ converges by the Alternating Series Test
9. $\sum_{n=1}^{\infty} \frac{1}{n} \cos n\pi$ converges by the Alternating Series Test.
10. $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{n\sqrt{n}}$ converges by the Alternating Series Test.
11. $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{3n+2}{n+10}$ fails the Alternating Series Test: terms do not decrease.
12. For $\sum_{k=1}^{\infty} (-1)^k \frac{5}{k^2}$ **:** $s_3 = -\frac{255}{36}$; $|s_3 - S| < \frac{5}{16}$.
13. For $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k!}$ **:** $s_4 = \frac{5}{8}; |s_4 - S| < \frac{1}{120}$.

14. $\left|\sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} - S\right| < 0.05$ for $k = 19$.

15. $\left|\sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} - S\right| < 0.005$ for $k = 199$.

16. $\left|\sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} - S\right| < 0.0001$ for $k = 9,999$.

Positive and Negative Term Series: Absolute and Conditional Convergence

Answers

1. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^{1/4}}$ converges conditionally
2. $\sum_{n=1}^{\infty} \frac{(-1)^{n}}{n^{4}}$ converges absolutely
3. $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{3k}{2^{k}}$ converges absolutely
4. $\sum_{k=1}^{\infty} \frac{(-1)^{k+1} k}{2k^{2}+2}$ converges conditionally
5. $\sum_{k=1}^{\infty} \frac{(-4)^{k+1}}{7k^{2}}$ is divergent (by nth term divergence test)
6. $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^{7/2}}$ converges absolutely
7. $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k}{3k^{2}+k}$ converges conditionally
8. $\sum_{n=1}^{\infty} \frac{(-1)^{n} e^{1/n}}{n^{3}}$ converges absolutely
9. $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{n\sqrt{n}}$ converges absolutely
10. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{e^{n}}$ converges absolutely
11. $\sum_{n=1}^{\infty} \frac{(-2)^{n}}{n^{2}}$ diverges (by nth term divergence test)
12. $\sum_{n=1}^{\infty} (-1)^{n} \frac{n}{(n+1)\sqrt{n}}$ converges conditionally (Use Limit Comparison Test with $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ to show absolute series divergence and Alternating Series Test to show conditional convergence.)

13. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n\sqrt{n^2+2}}$ converges absolutely (Compare absolute series with $\sum_{n=1}^{\infty} \frac{1}{n^2}$)

14. $\sum_{n=1}^{\infty} (-1)^n \frac{2n^2}{n^3+1}$ converges conditionally (Ratio test not conclusive for absolute series, try Integral Test to show divergence)

15. $\sum_{k=1}^{\infty} (-1)^k \frac{\ln k}{\sqrt{k}}$ converges conditionally.

Positive and Negative Term Series: The Ratio and Root Tests for Absolute Convergence

Answers

1. $\sum_{n=1}^{\infty}(-1)^k\frac{1}{2^{k-1}}$ converges absolutely by the Ratio Test:

$$\lim_{k\to\infty}\frac{\left|(-1)^{k+1}\frac{1}{2^{k+1-1}}\right|}{\left|(-1)^k\frac{1}{2^{k-1}}\right|}=\lim_{k\to\infty}\frac{\frac{1}{2^k}}{\frac{1}{2^{k-1}}}=\lim_{k\to\infty}\frac{2^{k-1}}{2^k}=\frac{1}{2}<1$$: Note that $\sum_{k=1}^{\infty}(-1)^k\frac{1}{2^{k-1}}=\sum_{k=1}^{\infty}-\left(-\frac{1}{2}\right)^{k-1}$ is

a convergent alternating geometric series.

2. $\sum_{k=1}^{\infty}\frac{(-1)^k}{k!}$ converges absolutely by the Ratio Test: $\lim_{k\to\infty}\frac{\left|\frac{(-1)^{k+1}}{(k+1)!}\right|}{\left|\frac{(-1)^k}{k!}\right|}=\lim_{k\to\infty}\frac{k!}{(k+1)!}=\lim_{k\to\infty}\frac{1}{k}=0<1$

.

3. $\sum_{n=1}^{\infty}\frac{(2n)!}{(-4)^n}=\sum_{n=1}^{\infty}(-1)^n\frac{(2n)!}{4^n}$ diverges: By the Ratio Test

$$\lim_{n\to\infty}\frac{\left|\frac{(2(n+1))!}{(-4)^{n+1}}\right|}{\left|\frac{(2n)!}{(-4)^n}\right|}=\lim_{n\to\infty}\frac{4^n}{4^{n+1}}\frac{(2n+2)!}{(2n)!}=\lim_{n\to\infty}\frac{1}{4}(2n+2)(2n+1)=\infty$$, means the series does not

converge absolutely (diverges). The Alternating Series test cannot be used because the series fails the requirement that $a_k>a_{k+1}$. Using the nth term divergence test shows that

$\lim_{n\to\infty}\frac{(2n)!}{4^n}=\infty$; and the series diverges.

4. $\sum_{n=1}^{\infty}\frac{(-1)^n(1.5)^n}{n^4}$ is not absolutely convergent by the Ratio Test; it diverges by the nth term

test for divergence. $\lim_{n\to\infty}\frac{\left|\frac{(-1)^{n+1}(1.5)^{n+1}}{(n+1)^4}\right|}{\left|\frac{(-1)^n(1.5)^n}{n^4}\right|}=\lim_{n\to\infty}\frac{(1.5)^{n+1}}{(1.5)^n}\frac{n^4}{(n+1)^4}=\lim_{k\to\infty}1.5\frac{1}{(1+1/n)^4}=1.5>1$.

5. $\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{2k+1}$ conditionally converges: $\lim_{k\to\infty}\frac{\left|\frac{(-1)^{k+2}}{2k+3}\right|}{\left|\frac{(-1)^{k+1}}{2k+1}\right|}=\lim_{k\to\infty}\frac{2k+1}{2k+3}=\lim_{k\to\infty}\frac{1+1/2k}{1+3/2k}=1$, means

Ratio Test is inconclusive. Using the Integral Test on $\sum_{k=1}^{\infty}\left|\frac{(-1)^{k+1}}{2k+1}\right|$ shows divergence. Use of the Alternating Series Test on $\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{2k+1}$ shows convergence.

6. $\sum_{n=1}^{\infty}(-1)^n\frac{n^2}{2^n}$ converges absolutely:

$$\lim_{n\to\infty}\frac{\left|(-1)^{n+1}\frac{(n+1)^2}{2^{n+1}}\right|}{\left|(-1)^n\frac{n^2}{2^n}\right|}=\lim_{n\to\infty}\frac{2^n}{2^{n+1}}\frac{(n+1)^2}{n^2}=\lim_{n\to\infty}\frac{1}{2}\left(\frac{n+1}{n}\right)^2=\frac{1}{2}<1.$$

7. $\sum_{n=1}^{\infty}(-1)^n\frac{3^{n-1}}{n^4}$ diverges : $\lim_{n\to\infty}\frac{(-1)^{n+1}\frac{3^n}{(n+1)^4}}{\left|(-1)^n\frac{3^{n-1}}{n^4}\right|}=\lim_{n\to\infty}\frac{3^n}{3^{n-1}}\frac{n^4}{(n+1)^4}=\lim_{n\to\infty}3\left(\frac{n}{n+1}\right)^4=3>1$;

nth term divergence test yields $\lim_{n\to\infty}\frac{3^{n-1}}{n^4}=\infty$ (L'Hopital's rule applied multiple times).

8. $\sum_{n=1}^{\infty}(-1)^n\frac{n+1}{3^n}$ converges absolutely by the nth Root Test:

$\lim_{n\to\infty}\sqrt[n]{\left|(-1)^n\frac{n+1}{3^n}\right|}=\lim_{n\to\infty}\frac{(n+1)^{1/n}}{3}=\frac{1}{3}<1$. Note: For $\lim_{n\to\infty}\frac{(n+1)^{1/n}}{3}$,

$$\lim_{n\to\infty}y=\lim_{n\to\infty}(n+1)^{1/n}\Rightarrow\lim_{n\to\infty}\ln y=\lim_{n\to\infty}\frac{\ln(n+1)}{n}=\lim_{n\to\infty}\frac{1/n}{1}=0\Rightarrow y=e^0=1.$$

9. $\sum_{n=1}^{\infty}\frac{(-1)^n}{\left[\ln(k+1)\right]^n}$ converges absolutely: $\lim_{n\to\infty}\sqrt[n]{\left|\frac{(-1)^n}{\left[\ln(n+1)\right]^n}\right|}=\lim_{n\to\infty}\frac{1}{\ln(n+1)}=0<1$.

10. $\sum_{n=2}^{\infty}(-1)^{n+1}\frac{n}{n^2-1}$ is conditionally convergent: $\lim_{n\to\infty}\sqrt[n]{\left|(-1)^{n+1}\frac{n}{n^2-1}\right|}=\left[\frac{\lim_{n\to\infty}n^{1/n}}{\lim_{n\to\infty}(n^2-1)^{1/n}}\right]=\frac{1}{1}=1$,

means not conclusive by Root Test. By Comparison Test for $\sum_{n=2}^{\infty}\left|(-1)^{n+1}\frac{n}{n^2-1}\right|$,

$\sum_{n=2}^{\infty}\frac{n}{n^2-1} > \sum_{n=2}^{\infty}\frac{n}{n^2} = \sum_{n=2}^{\infty}\frac{1}{n}$. Since $\sum_{n=2}^{\infty}\frac{1}{n}$ diverges, so does $\sum_{n=2}^{\infty}\frac{n}{n^2-1}$. The Alternating Series Test shows that $\sum_{n=2}^{\infty}(-1)^{n+1}\frac{n}{n^2-1}$ converges.

11. $\sum_{n=2}^{\infty}\frac{(-1)^n}{\ln(2n)}$ is conditionally convergent: $\lim_{n\to\infty}\sqrt[n]{\left|\frac{(-1)^n}{\ln(2n)}\right|} = \lim_{n\to\infty}\frac{1}{\left[\ln(2n)\right]^{1/n}} = 1$, means the Root Test is inconclusive.

($y = \left[\ln(2n)\right]^{1/n} \Rightarrow \lim_{n\to\infty}\ln y = \lim_{n\to\infty}\frac{\ln(\ln(2n))}{n} = \lim_{n\to\infty}\frac{(1/\ln(2n))\cdot(2/2n)}{1} = \lim_{n\to\infty}\frac{1}{n\ln(2n)} = 0$.

$\Rightarrow \lim_{n\to\infty} y = e^0 = 1 = \lim_{n\to\infty}(\ln(2n))^{1/n}$). By the Comparison Test $\sum_{n=2}^{\infty}\left|\frac{(-1)^n}{\ln(2n)}\right| > \sum_{n=2}^{\infty}\frac{1}{n}$, and since $\sum_{n=2}^{\infty}\frac{1}{n}$ is a Harmonic series and diverges, then $\sum_{n=2}^{\infty}\left|\frac{(-1)^n}{\ln(2n)}\right|$ diverges. By the Alternating Series Test, $\sum_{n=2}^{\infty}\frac{(-1)^n}{\ln(2n)}$ converges.

12. $\frac{2}{1.3} - \left(\frac{3}{2.4}\right)^2 + \left(\frac{4}{3.5}\right)^3 - \left(\frac{5}{4.6}\right)^4 + \cdots = \sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{n+1}{1.3+1.1(n-1)}\right)^n$ converges absolutely by the nth Root Test: $\lim_{n\to\infty}\sqrt[n]{\left|(-1)^{n-1}\left(\frac{n+1}{1.3+1.1(n-1)}\right)^n\right|} = \lim_{n\to\infty}\left(\frac{n+1}{1.3+1.1(n-1)}\right) = \frac{1}{1.1} < 1$.

13. $\sum_{n=1}^{\infty}(-1)^n\left(\frac{4n+2}{19n-1}\right)^{2n}$ converges absolutely:

$$\lim_{n\to\infty}\sqrt[n]{\left|(-1)^n\left(\frac{4n+2}{19n-1}\right)^{2n}\right|} = \lim_{n\to\infty}\left(\frac{4n+2}{19n-1}\right)^2 = \lim_{n\to\infty}\left(\frac{4+2/n}{19-1/n}\right)^2 = \left(\frac{4}{19}\right)^2 < 1$$

14. $\sum_{n=1}^{\infty}(-1)^n\left(\frac{n^2-1}{n^2+1}\right)^n$ diverges : $\lim_{n\to\infty}\sqrt[n]{\left|(-1)^n\left(\frac{n^2-1}{n^2+1}\right)^n\right|} = \lim_{n\to\infty}\left(\frac{n^2-1}{n^2+1}\right) = 1$, means the Root Test is inconclusive. The conditions for the Alternating Series Test are not satisfied. The nth Term Divergence Test shows $\lim_{n\to\infty}\left(\frac{n^2-1}{n^2+1}\right)^n = 1 > 0$ which means $\sum_{n=1}^{\infty}(-1)^n\left(\frac{n^2-1}{n^2+1}\right)^n$ diverges.

15. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{(-1)^n \ln n}{n}\right)^n$ converges absolutely by the Root Test:

$$\lim_{n\to\infty}\sqrt[n]{\left|\left(\frac{(-1)^n \ln n}{n}\right)^n\right|}=\lim_{n\to\infty}\left(\frac{\ln n}{n}\right)=\lim_{n\to\infty}\left(\frac{1/n}{1}\right)=0$$

Summary of Convergence Tests

Answers

1. $\sum_{n=3}^{\infty}\frac{5}{n-2}$ diverges: $f(x)=\frac{5}{x-2}$ is continuous, positive, and decreasing (

$f'(x)=-\frac{5}{(x-2)^2}<0,\ x\geq 3$):

$$\int_3^{\infty}\frac{5}{x-2}dx=\lim_{p\to\infty}\int_3^p\frac{5}{x-2}dx=5\cdot\lim_{p\to\infty}\left[\ln(x-2)\right]_3^p=5\cdot\lim_{p\to\infty}\left[\ln(p-2)-0\right]=\infty$$

2. $\sum_{n=2}^{\infty}\frac{\ln n}{n^2}$ diverges: $f(x)=\frac{\ln x}{x^2},\ x\geq 2$ is continuous, positive, and decreasing (

$f'(x)=-\frac{x(2\ln x-1)}{x^4}<0,\ x\geq 2$).

$$\int_1^{\infty}f(x)dx=\lim_{p\to\infty}\int_2^p\frac{\ln x}{x^2}dx=\lim_{p\to\infty}\left[\frac{x^3}{3}(\ln x-\frac{1}{3})\right]_2^p=\infty\,.$$

3. $\sum_{n=1}^{\infty}\frac{1}{2n^2+n+5}$ converges by Comparison test using $\sum_{n=1}^{\infty}\frac{1}{2n^2}$, a convergent p-series.

4. $\sum_{k=1}^{\infty}\frac{k^3+4k^2+1}{3k^6+2k^4}$ converges by the Limit Comparison Test: $\lim_{n\to\infty}\frac{\frac{k^3+4k^2+1}{3k^6+2k^4}}{1/3k^3}=\frac{1}{3}$ and using

$\sum_{k=1}^{\infty}\frac{1}{k^3}$, which is a convergent p-series.

5. $\sum_{n=1}^{\infty}\frac{n!}{n^n}$ converges by the Ratio Test:

$$\lim_{n\to\infty}\frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}}=\lim_{n\to\infty}\left(n+1\frac{n^n}{(n+1)(n+1)^n}\right)=\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n=\lim_{n\to\infty}\left(\frac{1}{1+1/n}\right)^n=\frac{1}{e}\,.$$

6. $\sum_{n=1}^{\infty} n\left(\frac{2}{3}\right)^n$ converges by the Root Test: $\lim_{n\to\infty}\left(n\left(\frac{2}{3}\right)^n\right)^{1/n} = \left(\frac{2}{3}\right)\lim_{n\to\infty} n^{1/n} = \frac{2}{3}$

Note: $\lim_{n\to\infty} n^{1/n} = 1$: $y = \lim_{n\to\infty} n^{1/n} \Rightarrow \ln y = \lim_{n\to\infty} \frac{\ln n}{n} = \lim_{n\to\infty} \frac{1/n}{1} = 0 \Rightarrow y = e^0 = 1$

7. $\sum_{n=1}^{\infty} \frac{(-2)^n n}{7^{n+1}}$ converges absolutely by the Root or Ratio Test:

$$\lim_{n\to\infty} \sqrt[n]{\frac{(2)^n n}{7^{n+1}}} = \lim_{n\to\infty} \frac{2n^{1/n}}{7^{(1+1/n)}} = \frac{2}{7} < 1 \ ; \ \lim_{x\to\infty}\left[\frac{\left|\frac{(-2)^{n+1}(n+1)}{7^{n+2}}\right|}{\left|\frac{(-2)^n n}{7^{n+1}}\right|}\right] = \lim_{x\to\infty}\left[\frac{7^{n+1}}{7^{n+2}}\frac{2^{n+1}}{2^n}\frac{n+1}{n}\right] = \frac{2}{7} < 1$$

8. $\sum_{n=1}^{\infty} (-1)^n \frac{3n^4+5}{4n^9-3n}$ is absolutely convergent by the Limit Comparison Test using $\sum_{n=1}^{\infty} \frac{n^4}{n^9} = \sum_{n=1}^{\infty} \frac{1}{n^5}$ (convergent p-series).

9. $\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n^2-1}}$ is conditionally convergent: absolute divergence is shown using Limit Comparison Test with $\sum_{n=2}^{\infty} \frac{1}{n}$, and the series is convergent by the Alternating Series Test.

10. $\sum_{n=1}^{\infty} \frac{\cos n}{n^3}$ is absolutely convergent by Comparison Test: $\sum_{n=1}^{\infty} \left|\frac{\cos n}{n^3}\right| \le \sum_{n=1}^{\infty} \frac{1}{n^3}$ (convergent p-series)

11. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n^2}}$ is conditionally convergent: absolutely diverges because becomes a p-series with p<1, and is an Alternating Series that is convergent.

12. $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$ absolutely converges by Comparison Test with p-series $\sum_{n=1}^{\infty} \frac{1}{n^2}$

13. $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$ is conditionally convergent: Absolutely diverges by Comparison test

$\sum_{n=2}^{\infty} \frac{1}{\ln n} > \sum_{n=2}^{\infty} \frac{1}{n}$ (Harmonic series), and converges by Alternating Series Test.

14. $\sum_{n=1}^{\infty} \frac{(-1)^n 2^n}{2n}$ diverges by the Ratio or Root Tests.

15. $\sum_{n=1}^{\infty} \frac{(-1)^n n!}{3^n}$ diverges by the Root Test.

Power Series: Introduction and Convergence

Answers

1. $\sum_{n=1}^{\infty} nx^n$ converges by the Ratio Test if $|x|<1$ and diverges if $|x|>1$; $R_c=1$, and the interval of convergence is $(-1,\ 1)$, and does not include either endpoint.

2. $\sum_{n=1}^{\infty} n!x^n$ diverges by the Ratio Test for all $|x|>0$; $R_c=0$

3. $\sum_{n=1}^{\infty} \frac{x^n}{n}$ converges by the Ratio Test if $|x|<1$ and diverges if $|x|>1$; $R_c=1$, and the interval of convergence is $[-1,\ 1)$, and includes $x=-1$ but not $x=1$

4. $\sum_{n=1}^{\infty} \frac{x^{n/3}}{n!}$ converges by the Ratio Test for all x; $R_c=\infty$

5. $\sum_{n=1}^{\infty} \sqrt{n}(x-x_0)^n$ converges by the Ratio Test if $|x-x_0|<1$ and diverges if $|x-x_0|>1$; $R_c=1$, and the interval of convergence is $(-1+x_0,\ 1+x_0)$, and does not include either endpoint.

6. $\sum_{n=0}^{\infty} a_n x^n$ converges at $x=5$ means it converges for $|x|<5$; $\sum_{n=0}^{\infty} a_n x^n$ diverges at $x=-7$ it diverges for $|x|>7$. Therefore

   a. $\sum_{n=0}^{\infty} a_n$ converges because $x=1$ is in the interval of convergence.

   b. $\sum_{n=0}^{\infty} a_n 3^n$ converges because $x=3$ is in the interval of convergence.

   c. $\sum_{n=0}^{\infty} a_n (-8)^n$ diverges because $x=-8$ is in the divergence region.

   d. $\sum_{n=0}^{\infty} a_n 9^n$ diverges because $x=9$ is in the divergence region.

   e. $\sum_{n=0}^{\infty} a_n 6^n$ convergence or divergence is not clear since $x=6$ is not in defined regions.

7. $\sum_{n=0}^{\infty} \frac{n(x+4)^n}{5^{n+1}}$ converges by the Ratio Test for $\left|\frac{x+4}{5}\right|<1$ and diverges for $\left|\frac{x+4}{5}\right|>1$, $R_c=5$, and the interval of convergence is $(-9,\ 1)$, with divergence at both endpoints.

8. $\sum_{n=1}^{\infty} 3^n (x-2)^n$ converges by the Ratio Test for $\left|3(x-2)\right| < 1$ and diverges for $\left|3(x-2)\right| > 1$; $R_c = \frac{1}{3}$, and the interval of convergence is $(\frac{5}{3},\ \frac{7}{3})$, with divergence at both endpoints.

9. $\sum_{n=1}^{\infty} (x+4)^n$ converges by the Ratio or Root Test for $\left|x+4\right| < 1$ and diverges for $\left|x+4\right| > 1$; $R_c = 1$, and the interval of convergence is $(-5,\ -3)$, with divergence at both endpoints.

10. $\sum_{n=0}^{\infty} \frac{n!x^n}{2^n}$ diverges by the Ratio Test for any $\left|x\right| > 0$ and converges only at its center $x = 0$; $R_c = 0$.

11. $\sum_{n=0}^{\infty} \frac{(2x)^n}{n^2}$ converges by the Ratio Test if $2\left|x\right| < 1$ and diverges if $2\left|x\right| > 1$; $R_c = \frac{1}{2}$, and the interval of convergence is $[-\frac{1}{2},\ \frac{1}{2}]$, includes both endpoints.

12. $\sum_{n=0}^{\infty} \frac{x^n}{4n^2+3}$ converges by the Ratio Test if $\left|x\right| < 1$ and diverges if $\left|x\right| > 1$; $R_c = 1$, and the interval of convergence is $[-1,\ 1]$, includes both endpoints.

13. $\sum_{n=0}^{\infty} \frac{4^n x^{2n}}{n+3}$ converges by the Ratio Test if $\left|4x^2\right| < 1 \Rightarrow \left|x\right| < \frac{1}{2}$ and diverges if $\left|x\right| > \frac{1}{2}$; $R_c = \frac{1}{2}$, and the interval of convergence is $[-\frac{1}{2},\ \frac{1}{2})$; series converges at the left endpoint and diverges at the right.

14. $\sum_{n=1}^{\infty} \frac{5^n (x-3)^n}{n4^n}$ converges absolutely by the Ratio Test if $\left|\frac{5}{4}(x-3)\right| < 1 \Rightarrow \left|(x-3)\right| < \frac{4}{5}$ and diverges if $\left|\frac{5}{4}(x-3)\right| > 1$; $R_c = \frac{4}{5}$, and the interval of convergence is $[\frac{11}{5},\ \frac{19}{5})$, which only includes the endpoint at $x = \frac{11}{5}$

15. $\sum_{n=0}^{\infty} \frac{n(x+1)^{2n}}{7^n}$ converges absolutely by the Ratio Test for $\left|\frac{(x+1)^2}{7}\right| < 1 \Rightarrow \left|x+1\right| < \sqrt{7}$; $R_c = \sqrt{7}$, and the interval of convergence is $(-\sqrt{7}-1,\ \sqrt{7}-1)$; series diverges at endpoints.

Power Series: Function Representation and Operations

Answers

1. $f(x)=\dfrac{1}{1-2x^2}$ in domain $\left(-\dfrac{\sqrt{2}}{2},\dfrac{\sqrt{2}}{2}\right)$: $\displaystyle\sum_{n=0}^{\infty}2^n x^{2n}=\sum_{n=0}^{\infty}(2x^2)^n$ is a geometric series that converges if $\left|2x^2\right|<1$, $R_c=\dfrac{\sqrt{2}}{2}$

2. $\displaystyle\frac{x}{2-x}=\frac{x}{2}\cdot\frac{1}{1-x/2}=\frac{x}{2}\cdot\sum_{n=1}^{\infty}\left(\frac{x}{2}\right)^{n-1}=\sum_{n=1}^{\infty}\left(\frac{x}{2}\right)^{n}=\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^{n+1}$ for $\left|\dfrac{x}{2}\right|<1$, $R_c=2$

3. $\displaystyle\frac{1}{(1-x)^2}=\frac{d}{dx}\left(\frac{1}{1-x}\right)=\frac{d}{dx}\left(\sum_{n=0}^{\infty}x^n\right)=1+2x+3x^2+4x^3+\cdots=\sum_{n=0}^{\infty}(n+1)x^n$; or

   $\displaystyle\frac{1}{(1-x)^2}=\left(\frac{1}{1-x}\right)^2=\left(\sum_{n=0}^{\infty}x^n\right)\cdot\left(\sum_{n=0}^{\infty}x^n\right)=1+2x+3x^2+4x^3+\cdots=\sum_{n=0}^{\infty}(n+1)x^n$, for $|x|<1$, $R_c=1$.

4. $\displaystyle\frac{x^2}{(1-x)^3}=x^2\cdot\frac{d}{dx}\left(\frac{1}{2(1-x)^2}\right)=\frac{x^2}{2}\left(\sum_{n=1}^{\infty}n(n+1)x^{n-1}\right)=\sum_{n=0}^{\infty}\frac{(n+1)(n+2)}{2}x^{n+2}$

   $\displaystyle\frac{x^2}{(1-x)^3}=x^2\cdot\frac{1}{(1-x)^3}=x^2\left(\sum_{n=0}^{\infty}x^n\right)^3=x^2\sum_{n=0}^{\infty}\frac{(n+1)(n+2)}{2}x^n=\sum_{n=0}^{\infty}\frac{(n+1)(n+2)}{2}x^{n+2}$

5. $\displaystyle\frac{2x}{(1-x)^3}+\frac{3x^2}{(1-x)^4}=2x\left(\sum_{n=1}^{\infty}n(n+1)x^{n-1}\right)+3x^2\frac{d}{dx}\left[\frac{1}{3(1-x)^3}\right]$

   $\displaystyle=\left(\sum_{n=1}^{\infty}2n(n+1)x^n\right)+3x^2\left[\sum_{n=1}^{\infty}\frac{(n-1)n(n+1)x^{n-2}}{3}\right]=\sum_{n=1}^{\infty}n(n+1)^2x^n$

6. $\displaystyle\frac{12x}{7-2x^2}=\frac{12x}{7}\cdot\frac{1}{(1-2x^2/7)}=\frac{12x}{7}\cdot\sum_{n=1}^{\infty}\left(\frac{2x^2}{7}\right)^{n-1}=6\cdot\sum_{n=0}^{\infty}\left(\frac{2}{7}\right)^{n+1}x^{2n+1}$, with $|x|<\dfrac{\sqrt{14}}{2}$

7. $\displaystyle\frac{d}{dx}\ln(1+x^2)=\frac{2x}{1+x^2}\Rightarrow\ln(1+x^2)=\int\frac{2x}{1+x^2}dx=\int2x\sum_{n=0}^{\infty}\left(-x^2\right)^n dx=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2(n+1)}}{n+1}$; domain (-1,1), $R_c=1$.

8. $\displaystyle\tan^{-1}x=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1}\Rightarrow\int\tan^{-1}x\,dx=\int\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1}dx=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+2}}{(2n+2)(2n+1)}$; interval of convergence (domain) (-1,1), $R_c=1$

9. $\frac{d}{dx}\ln(1+x+x^2)=\frac{2x+1}{1+x+x^2}\Rightarrow \ln(1+x+x^2)=\sum_{n=0}^{\infty}2\left(\frac{4}{3}\right)^{n+1}\frac{1}{n+2}\left(x+\frac{1}{2}\right)^{n+2}$, $R_c=\frac{3}{4}$ and the interval of convergence of $\left[-\frac{5}{4},\frac{1}{4}\right)$

$$\int\frac{2x+1}{1+x+x^2}dx=\int\frac{2(x+1/2)}{(x+1/2)^2+3/4}dx=\int\frac{(8/3)(x+1/2)}{1+(4/3)(x+1/2)^2}dx=\int\sum_{n=0}^{\infty}2\left(\frac{4}{3}(x+\frac{1}{2})\right)^{n+1}dx$$

10. $\int x^2e^x dx=\int x^2\left[\sum_{n=0}^{\infty}\frac{x^n}{n!}\right]dx=\left[\sum_{n=0}^{\infty}x^2\int\frac{x^n}{n!}dx\right]=\left[\sum_{n=0}^{\infty}x^2\frac{x^{n+1}}{(n+1)n!}\right]=\sum_{n=0}^{\infty}\frac{x^{n+3}}{(n+1)n!}$, $R_c=\infty$

11. $\frac{1}{(1-rx)(1-sx)}=\left(\sum_{n=0}^{\infty}r^nx^n\right)\cdot\left(\sum_{n=0}^{\infty}s^nx^n\right)=\sum_{n=0}^{\infty}c_nx^n$, where $c_n=\sum_{k=0}^{n}r^ks^{n-k}$ and radius of convergence $R_c=\min(\frac{1}{r},\frac{1}{s})$

a. $r\neq s, r>0, s>0$; $\frac{1}{(1-rx)(1-sx)}=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}r^ks^{n-k}\right)x^n$

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}r^ks^{n-k}\right)x^n=1+(r+s)x+(s^2+rs+r^2)x^2+(s^3+rs^2+r^2s+r^3)x^3+\cdots$$

b. $r=s, r>0, s>0$: $\frac{1}{(1-rx)(1-sx)}=\sum_{n=0}^{\infty}(n+1)r^nx^n$, with $R_c=\frac{1}{s}=\frac{1}{r}$

Introduction to Taylor and Maclaurin Series

Answers

1. see Table in the concept: $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

2. $f(x) = \sin 2x = M(x) = \sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n+1}}{(2n+1)!} = (2x) - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \frac{(2x)^7}{7!} + \cdots$

3. $\tan x = M(x) = x + 2\frac{x^3}{3!} + 16\frac{x^5}{5!} + 272\frac{x^7}{7!} + \cdots = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \cdots$

4. $T(x) = 1 + 2\left(x - \frac{\pi}{4}\right) + \frac{4}{2!}\left(x - \frac{\pi}{4}\right)^2 + \frac{12}{3!}\left(x - \frac{\pi}{4}\right)^3 + \frac{36}{4!}\left(x - \frac{\pi}{4}\right)^4 + \cdots$

5. $f(x) = \sqrt{1+x} = M(x) = 1 + \frac{1}{2}x + \left(-\frac{1}{4}\frac{x^2}{2!} + \frac{3}{8}\frac{x^3}{3!} - \frac{5}{16}\frac{x^4}{4!} + \cdots\right) = 1 + \frac{1}{2}x + \sum_{n=2}^{\infty} (-1)^{n-1}\left(\frac{2n-3}{2^n}\right)\frac{x^n}{n!}$

6. $\sqrt{(1+x+x^2)} = \left(\frac{4}{3}\right)^{-1/2} + \left(\frac{4}{3}\right)^{1/2}\frac{(x+1/2)^2}{2!} + 3\left(\frac{4}{3}\right)^{3/2}\frac{(x+1/2)^2}{4!} + 45\left(\frac{4}{3}\right)^{5/2}\frac{(x+1/2)^2}{6!} + \cdots$

7. $\cos^2 x = M(x) = \frac{1}{2} + \frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n(2x)^{2n}}{(2n)!} = \frac{1}{2} + \sum_{n=0}^{\infty}\frac{(-1)^n(2x)^{2n}}{2\cdot(2n)!}$

8. $\frac{x}{e^x} = xe^{-x} = M(x) = x\sum_{n=0}^{\infty}\frac{(-x)^n}{n!} = x\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{n!} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{n!}$

9. $e^{\cos x} = M(x) = \sum_{n=0}^{\infty}\frac{(\cos x)^n}{n!}$

10. $(1+x)^{\alpha} = M(x) = 1 + \alpha x + \alpha(\alpha-1)\frac{x^2}{2!} + \alpha(\alpha-1)(\alpha-2)\frac{x^3}{3!} + \cdots$

11. $\ln(3+9x) = M(x) = \ln 3 + 3x - \frac{x^2}{2!} + \frac{2x^3}{3!} - \frac{6x^4}{4!} + \cdots (-1)^{n+1}\frac{x^n}{n} + \cdots;\ n \geq 2$

12. $$\lim_{x\to 0}\frac{\sin kx}{x}=\lim_{x\to 0}\left[\frac{k}{kx}\sum_{n=0}^{\infty}(-1)^n\frac{(kx)^{2n+1}}{(2n+1)!}\right]=\lim_{x\to 0}\left[\sum_{n=0}^{\infty}(-1)^n\frac{k(kx)^{2n}}{(2n+1)!}\right]=k$$

13. $$e^{-x}\cos x=-e^{-\pi/2}(x-\frac{\pi}{2})+2e^{-\pi/2}\frac{(x-\frac{\pi}{2})^2}{2!}-2e^{-\pi/2}\frac{(x-\frac{\pi}{2})^3}{3!}+4e^{-\pi/2}\frac{(x-\frac{\pi}{2})^5}{5!}-8e^{-\pi/2}\frac{(x-\frac{\pi}{2})^6}{6!}$$

$$+8e^{-\pi/2}\frac{(x-\frac{\pi}{2})^7}{7!}+0\square 16e^{-\pi/2}\frac{(x-\frac{\pi}{2})^8}{8!}+\square\square\square$$

14. For $f(x)=\sum_{n=1}^{\infty}\frac{(-1)^n(x-7)^{n+1}}{n!}$, $f^{(6)}(7)$ occurs at $n=5$. Therefore

$$f^{(6)}(7)\frac{(x-7)^6}{6!}=\frac{(-1)^5(x-7)^{5+1}}{5!}\Rightarrow f^{(6)}(7)=-\frac{6!}{5!}=6$$

15. $$\int\frac{e^x-1}{x}dx=\int\frac{1}{x}\left[-1+\sum_{n=0}^{\infty}\frac{x^n}{n!}\right]dx=\int\frac{1}{x}\left[\sum_{n=1}^{\infty}\frac{x^n}{n!}\right]dx=\int\left[\sum_{n=1}^{\infty}\frac{x^{n-1}}{n!}\right]dx=\sum_{n=1}^{\infty}\frac{x^n}{n\square n!}+C$$

$(R_c=\infty)$

Taylor and Maclaurin Polynomials: Series Truncation Error

Answers

1. $f(x)=\sqrt{x}\approx 1+\frac{1}{2}(x-1)-\frac{1}{8}(x-1)^2$

2. $f(x)=e^{3x}\approx 1+3x+9\frac{x^2}{2!}+27\frac{x^3}{2!}$

3. $f(x)=\ln 4x\approx (x-1)-\frac{(x-1)^2}{2}+\frac{(x-1)^3}{3}-\frac{(x-1)^4}{4}$

4. $f(x)=1+x+x^2+x^3+x^4\approx 1-2(x-1)+4(x-1)^2-3(x-1)^3+(x-1)^4$

5. $f(x)=\frac{4}{3x-7}\approx -\frac{4}{7}-\frac{4\cdot 3}{7^2}x-\frac{4\cdot 3^2}{7^3}x^2-\frac{4\cdot 3^3}{7^4}x^3-\frac{4\cdot 3^4}{7^5}x^4-\frac{4\cdot 3^5}{7^6}x^5$

6. $f(x)=\cos\left(\frac{2}{3}x\right)\approx\frac{1}{2}-\left(\frac{2}{3}\right)\left(\frac{\sqrt{3}}{2}\right)(x-\pi/2)-\left(\frac{2}{3}\right)^2\left(\frac{1}{2}\right)(x-\pi/2)^2+\left(\frac{2}{3}\right)^3\left(\frac{\sqrt{3}}{2}\right)(x-\pi/2)^3+\cdots$

$$=\sum_{n=0}^{2}\left[(-1)^n\frac{1}{2}\left(\frac{2}{3}\right)^{2n}\frac{(x-\pi/2)^{2n}}{(2n)!}+(-1)^{n+1}\frac{\sqrt{3}}{2}\left(\frac{2}{3}\right)^{2n+1}\frac{(x-\pi/2)^{2n+1}}{(2n+1)!}\right]$$

7. a. $h(x)=3-2(x-5)+7\frac{(x-5)^2}{2!}-3\frac{(x-5)^3}{3!}$

   b. $h(4.8)\approx 3-2(-0.2)+7\frac{(-0.2)^2}{2!}-3\frac{(-0.2)^3}{3!}=3.544$

8. $f(x)=\sqrt{1+x}\approx 1-x+\frac{x^2}{8}-\frac{5x^2}{128}$, with $\left|R_n\right|=\frac{105}{32}\left|1-0.9\right|^{9/2}\frac{0.1^5}{5!}=4.39\times 10^{-7}$

9. $e^x\approx\sum_{n=0}^{5}\frac{x^n}{n!}=2.71666$ is the 5th degree Taylor polynomial centered at $x_0=0$

   $|R_n(x)|=|R_5(x)|\le\frac{f^{(6)}(z)}{(n+1)!}|x|^{n+1}=\frac{e^1}{6!}|1|^6=0.00377$.

   Calculator value is 2.71828; $2.71828-2.71666=0.00162<0.00377$

10. $\left|R_n\right|=\frac{f^{(n)}(z)}{(n+1)!}|x|^{n+1}=\frac{e^{½}}{(n+1)!}(1/2)^{n+1}\le 10^{-3}\Rightarrow n=4,\ \left|R_n\right|=0.00043$

11. $\left|R_n\right|=\frac{f^{(n)}(z)}{(n+1)!}|x|^{n+1}=\frac{e^{½}}{(n+1)!}(1/2)^{n+1}\le 10^{-6}\Rightarrow n=7,\ \left|R_n\right|=1.6\times 10^{-7}$

12. $\left|R_n\right|=\frac{f^{(n)}(z)}{(n+1)!}|x|^{n+1}=\frac{e^2}{(n+1)!}(2)^{n+1}\le 10^{-6}\Rightarrow n=14,\ \left|R_n\right|=1.85\times 10^{-7}$ TBS

13. Estimate value of $\sin(2 radians)$ using a 5th degree Taylor polynomial:

a. Centered at $\frac{\pi}{2}$: $\sin x \approx 1 - \frac{(x-\pi/2)^2}{2!} + \frac{(x-\pi/2)^4}{4!}$; $\sin 2 \approx 0.90930608$,

$|R_n| = 8.653 \times 10^{-6}$. $|\sin(2) - 0.90930608| = 8.653 \times 10^{-6} < R_n$

b. Centered at 0: $\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!}$; $\sin 2 \approx 0.933333333$, $|R_n| = 1.27 \times 10^{-2}$

$|\sin(2) - 0.933333333| = 2.40 \times 10^{-2} < R_n$.

14. $f(x) = e^{(x^5 - x)} \approx 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!}$

$$|R_n| = \frac{f^{(n)}(z)}{(n+1)!} |x|^{n+1} = \frac{2^{n+1}}{(n+1)!} \le 10^{-2} \Rightarrow n = 7, \ |R_n| = 6.35 \times 10^{-3}$$

Taylor Series Calculations: Choosing Centers

Answers

1. With center at $x_0 = 4$: $\sqrt{x} \approx T_1(x) = 2 + \frac{1}{4}(x-4) \Rightarrow \sqrt{4.1} \approx 2 + \frac{1}{4}(4.1-4) = 2 + \frac{1}{40} = 2.025$.

   Calculator value: $\sqrt{4.1} = 2.024845673$; error magnitude is 1.54×10^{-4}

   $\sqrt{0.1} \approx 2 + \frac{1}{4}(0.1-4) = 2 - \frac{3.9}{40} = 1.025$ is a very poor estimate of $\sqrt{0.1} = 0.316227766...$

   The center of $x_0 = 4$ is not close enough to 0.1; the alternative is to use a higher degree polynomial.

2. $\ln(0.9) = \ln\left[1 + (-0.1)\right] \approx 0.105\overline{3} = M_3(-0.1)$; Calculator value: $\ln(0.9) = -0.105360516$

   $$\left|R_n(x)\right| = \left|\frac{x^{n+2}}{n+2}\right| < 5 \times 10^{-5}; \ \left|R_2(-0.1)\right| = \left|\frac{(-0.1)^4}{4}\right| = 2.5 \times 10^{-5}$$

3. With center at $x_0 = \frac{\pi}{4}$, $\sin(0.8) \approx T_2(0.8) = 0.717356 = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}(0.8 - \frac{\pi}{4}) - \frac{\sqrt{2}}{2}\frac{(0.8 - \frac{\pi}{4})^2}{2!}$,;

   Maclaurin polynomial is easier to compute, but takes more terms for the same result:

   $$\sin(0.8) \approx M_7(0.8) = 0.717356 = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$

4. With center at $x_0 = 2\pi$, $\sin(6) \approx T_3(6) = (x-2\pi) - \frac{(x-2\pi)^3}{3!} = -0.2832 + 0.0038 = -0.2794$;

   Calculator value: $\sin(6) = -0.279415498$. Note: $\sin(6) = \sin(6-2\pi) \approx M_3(6-2\pi) = T_3(6)$

5. $\frac{1}{9^3} = \frac{1}{(10-1)^3} = \frac{10^{-3}}{(1-1/10)^3} \approx M_4(0.1) = 0.0013715$; Calculator value: $\frac{1}{9^3} = 0.001371742$.

   $$\frac{1}{(1-x)^3} = \frac{d^2y}{dx^2}\left[\frac{1}{2(1-x)}\right]; \ \frac{10^{-3}}{(1-1/10)^3} = 10^{-3}\sum_{n=2}^{\infty}\frac{n(n-1)}{2}\left(\frac{1}{10}\right)^{n-2} = 10^{-3}\sum_{n=0}^{\infty}\frac{(n+2)(n+1)}{2}\left(\frac{1}{10}\right)^{n}$$

6. With center at $x_0 = 9$, $n = 1$, $\sqrt{x} \approx T_1(x) = 3 + \frac{1}{6}(x-9) \Rightarrow \sqrt{10} \approx 3 + \frac{10-9}{6} = 3.1\overline{6}$;

   $$\left|R_1(10)\right| \le \left|-\frac{1}{4 \square 9^{3/2}} \square \frac{(10-9)^2}{2!}\right| = \frac{1}{216} = 0.00462963 < 0.005;$$

   Calculator value: $\sqrt{10} = 3.16227766$

7. A Maclaurin polynomial for $\sqrt{x}$ cannot be used to estimate the value of $\sqrt{0.9}$ because derivatives at $x_0 = 0$ are not defined. The value can be estimated by using either a Maclaurin polynomial for a related function which has defined derivatives at $x_0 = 0$, e.g. $f(x) = \sqrt{1+x}$, or a Taylor polynomial for $\sqrt{x}$ centered at $x_0 = 1$

8. Calculator value: $\sqrt{99} = 9.949874371$

With center at $x_0 = 100$:

$\sqrt{x} \approx T_1(x) = 10 + \frac{1}{20}(x-100) \Rightarrow \sqrt{99} \approx 10 - 0.05 = 9.95$; error of 1.26×10^{-4}

$\sqrt{x} \approx T_2(x) = 10 + \frac{1}{20}(x-100) - \frac{1}{4000}\frac{(x-100)^2}{2!} \Rightarrow \sqrt{99} \approx 10 - \frac{1}{20} - \frac{1}{8000} = 9.949875$; error of 6.29×10^{-7}

$|R_2(99)| \le \left| -\frac{3}{8\cdot 99^{5/2}}\cdot\frac{(99-100)^3}{3!} \right| = 6.41\times10^{-7}$.

9. $\ln(1-x) = \sum_{n=0}^{\infty}(-1)\frac{x^{n+1}}{n+1} = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} + \cdots$ with the convergence interval $[-1,\ 1)$; using $x = -4$ is outside the interval of convergence and produces a divergent series.

10. With center $x_0 = 0: e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$, for all x. 5 decimal places means error $\le 5\times10^{-6}$

$e^{-0.2} \approx 0.81873 = M_4(-0.2) = 1 + \frac{(-0.2)}{1} + \frac{(-0.2)^2}{2!} + \frac{(-0.2)^3}{3!} + + \frac{(-0.2)^4}{4!}$

Calculator value: $e^{-0.2} = 0.818730753$; $|R_4(-0.2)| = 2.7\times10^{-6}$

11. Center at $x_0 = 0$; The Taylor polynomial $\cos(x) = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}$ is an alternating series.

The error is less than the next term: $\left|(-1)^{n+1}\frac{x^{2(n+1)}}{(2n+2)!}\right| < 10^{-20} \Rightarrow n = 3$

Calculator value: $\cos(0.01) = 0.99995$

$\cos(0.01) \approx M_6(0.01) = \sum_{n=0}^{3}(-1)^n\frac{x^{2n}}{(2n)!}$. Note $M_2(0.01) = 0.99995$

12. Because the Taylor center at $x_0 = \frac{\pi}{4} \approx 0.785$ is closer to 0.8 than the Maclaurin center $x_0 = 0$, the two-term Taylor polynomial approximation should give a more accurate estimate of $\tan(0.8)$. Calculator value: $\tan(0.8) = 1.029638557$

$\tan(x) \approx T_1(x) = 1 + 2(x - \frac{\pi}{4}) \Rightarrow \tan(0.8) \approx 1 + 2(0.8 - \frac{\pi}{4}) = 1.0292$; error magnitude $\approx 4.4\times10^{-4}$.

$\tan(x) \approx M_3(x) = x + 2\frac{x^3}{3!} \Rightarrow \tan(0.8) \approx 0.8 + 2\frac{0.8^3}{3!} = 0.9707$; error magnitude $\approx 5.9\times10^{-2}$

13. With center at $x_0 = \frac{\pi}{3}$ (close to 1.1):

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n \sqrt{3}}{2(2n)!}\left(x - \frac{\pi}{3}\right)^{2n} + \sum_{n=0}^{\infty} \frac{(-1)^n}{2(2n+1)!}\left(x - \frac{\pi}{3}\right)^{2n+1}$$

$$\sin(1.1) \approx T_3(1.1) = 0.86481823 + 0.02638773 \approx 0.8912$$

Calculator value: $\sin(1.1) = 0.89120736$

Calculations with series: Binomial fractional powers, integrals, and differential equations

Answers

1. $$(9-x)^{-1/2}=3(1-x/9)^{-1/2}=3\left[1+\left(-\frac{1}{2}\right)\left(-\frac{x}{9}\right)+\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\frac{1}{2!}\left(-\frac{x}{9}\right)^2+\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\frac{1}{3!}\left(-\frac{x}{9}\right)^3+\ldots\right]$$

$$=3\left[1+\frac{1}{18}x+\frac{1\cdot3}{2!18^2}x^2+\frac{1\cdot3\cdot5}{3!18^3}x^3+\ldots\frac{1\cdot3\cdot5\ldots\cdot(2n-1)}{n!18^n}x^n+\ldots\right],\quad\left|\frac{x}{9}\right|<1$$

2. $$\frac{1}{\sqrt{1+x}}=(1+x)^{-1/2}=1+\left(-\frac{1}{2}\right)x+\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\frac{x^2}{2!}+\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\frac{x^3}{3!}+\cdots$$

$$=1-\frac{1}{2}x+\frac{1\cdot3}{2!2^2}x^2-\frac{1\cdot3\cdot5}{3!2^3}x^3+\ldots(-1)^n\frac{1\cdot3\cdot5\ldots\cdot(2n-1)}{n!2^n}x^n+\ldots,\quad|x|<1$$

3. $$\frac{1}{(2-x)^2}=\frac{1}{4}(1-x/2)^{-2}=\frac{1}{4}\left[1+(-2)\left(-\frac{x}{2}\right)+(-2)(-3)\frac{1}{2!}\left(-\frac{x}{2}\right)^2+(-2)(-3)(-4)\frac{1}{3!}\left(-\frac{x}{2}\right)^3+\cdots\right]$$

$$=\frac{1}{4}\left[1+\frac{2}{2^1}x+\frac{3}{2^2}x^2+\frac{4}{2^3}x^3+\ldots+\frac{(n+1)}{2^n}x^n+\ldots\right]\text{ for }|x|<2$$

4. $$\left(1-\frac{1}{x}\right)^{1/2}=1-\left(\frac{1}{2}\right)v+\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\frac{v^2}{2!}-\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\frac{v^3}{3!}+\cdots$$

$$=1-\left(\frac{1}{2}\right)v-\frac{v^2}{2^2 2!}-\frac{3v^3}{2^3 3!}-\frac{3\cdot5v^4}{2^4 4!}+\cdots,\text{ where }v=\frac{1}{x},\quad|x|>1$$

5. $$\sqrt{(1+x+x^2)}=\left[\frac{3}{4}+\left(x+\frac{1}{2}\right)^2\right]^{1/2}=\frac{\sqrt{3}}{2}\left[1+\frac{3}{4}\left(x+\frac{1}{2}\right)^2\right]^{1/2}$$

$$=\frac{\sqrt{3}}{2}\left[1+\left(\frac{1}{2}\right)\left(\frac{3}{4}\right)v-\left(\frac{1}{2^2}\right)\left(\frac{3}{4}\right)^2\frac{v^2}{2!}+\left(\frac{3}{2^3}\right)\left(\frac{3}{4}\right)^3\frac{v^3}{3!}-\left(\frac{3\cdot5}{2^4}\right)\left(\frac{3}{4}\right)^4\frac{v^4}{4!}+\cdots\right]$$

where $v=\left(x+\frac{1}{2}\right)^2$, and $\left|\frac{3}{4}\left(x+\frac{1}{2}\right)^2\right|<1$, or $-\frac{\sqrt{3}}{2}-\frac{1}{2}<x<\frac{\sqrt{3}}{2}+\frac{1}{2}$

6. $\int_0^1 e^{x^2}dx \approx M_8(1) = \sum_{n=0}^{8} \frac{1^{2n+1}}{(2n+1)n!} \approx 1.462652$. $\int e^{x^2}dx = \int\left(\sum_{n=0}^{\infty} \frac{(x^2)^n}{n!}\right)dx = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)n!}$;

$\left|R_n\right| = \frac{x^{2n+3}}{(2n+3)n!} \le 5\times 10^{-7} \Rightarrow n = 8, \left|R_8\right| = 1.45\times 10^{-7}$

7. $\int \sin x^2 dx = \int\left(\sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{2n+1}}{(2n+1)}\right)dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+3}}{(4n+3)(2n+1)}$

8. $\int_0^1 \frac{\arctan x}{x}dx = \int_0^1\left(\frac{1}{x}\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}\right)dx = \sum_{n=0}^{\infty}\int_0^1 (-1)^n \frac{x^{2n}}{2n+1}dx = \left[\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)^2}\right]_0^1$

$\int_0^1 \frac{\arctan x}{x}dx = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)^2}$

9. $y = a_0 + a_0 x + \frac{a_0}{2!}x^2 + \frac{a_0}{3!}x^3 + \cdots = a_0\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right) = a_0 e^x$

10. $y = a_0 + a_0(-3x) + \frac{a_0}{2!}(-3x)^2 + \frac{a_0}{3!}(-3x)^3 + \cdots = a_0\left(1 + (-3x) + \frac{(-3x)^2}{2!} + \frac{(-3x)^3}{3!} + \frac{(-3x)^4}{4!} + \cdots\right) = a_0 e^{-3x}$

11. $y = a_0\left(1 + (x^2) + \frac{(x^2)^2}{2!} + \frac{(x^2)^3}{3!} + \frac{(x^2)^4}{4!} + \cdots\right) = a_0 e^{x^2}$

12. $y = a_0\left(1 + (-\frac{3}{2}x^2) + \frac{(-\frac{3}{2}x^2)^2}{2!} + \frac{(-\frac{3}{2}x^2)^3}{3!} + \frac{(-\frac{3}{2}x^2)^4}{4!} + \cdots\right) = a_0 e^{-3x^2/2}$

13. $y = a_0\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) + a_1\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right) = a_0 \cos x + a_1 \sin x$