Pascal’s Triangle and the Coefficients in the Expansion of Binomials

Lori Jordan
Kate Dirga
Learning Objectives

Here you’ll observe and use the connection between Pascal’s Triangle and expanded binomials to assist in expanding binomials.

Without multiplying \((x - 2)\) by itself five times, how could you expand \((x - 2)^5\)?

Pascal’s Triangle

Each row begins and ends with a one. Each “interior” value in each row is the sum of the two numbers above it. For example, \(2 + 1 = 3\) and \(10 + 10 = 20\). This pattern produces the symmetry in the triangle.

Another pattern that can be observed is that the row number is equal to the number of elements in that row. Row 1, for example has 1 element, 1. Row 2 has 2 elements, 1 and 1. Row 3 has 3 elements, 1, 2 and 1.

A third pattern is that the second element in the row is equal to one less than the row number. For example, in row 5 we have 1, 4, 6, 4 and 1.

Let’s continue the triangle to determine the elements in the 9th row of Pascal’s Triangle.

Following the pattern of adding adjacent elements to get the elements in the next row, we find that the eighth row is: 1 7 21 35 35 21 7 1

Now, continue the pattern again to find the 9th row: 1 8 28 56 70 56 28 8 1

Now, let’s expand the binomial \((a + b)^4\) and discuss the pattern within the exponents of each variable as well as the pattern found in the coefficients of each term.

\[
\begin{align*}
(a + b)(a + b)(a + b)(a + b) & \\
(a^2 + 2ab + b^2)(a^2 + 2ab + b^2) & \\
a^4 + 2a^3b + a^2b^2 + 2a^3b + 4a^2b^2 + 2ab^3 + a^2b^2 + 2ab^3 + b^4 & \\
a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 &
\end{align*}
\]
1. Take two binomials at a time and square them using \((a + b)^2 = a^2 + 2ab + b^2\)

2. Next, distribute each term in the first trinomial over each term in the second trinomial and collect like terms.

We can see that the powers of \(a\) start with 4 (the degree of the binomial) and decrease by one each term while the powers of \(b\) start with zero and increase by one each term. The number of terms is 5 which is one more than the degree of the binomial. The coefficients of the terms are 1 4 6 4 1, the elements of row 5 in Pascal’s Triangle.

Finally, let’s use what was discovered in the previous problem to expand \((x + y)^6\).

The degree of this expansion is 6, so the powers of \(x\) will begin with 6 and decrease by one each term until reaching 0 while the powers of \(y\) will begin with zero and increase by one each term until reaching 6. We can write the variables in the expansion (leaving space for the coefficients) as follows:

\[
\phantom{x^6} + \phantom{x^5}y + \phantom{x^4}y^2 + \phantom{x^3}y^3 + \phantom{x^2}y^4 + \phantom{x}y^5 + \phantom{y}^6
\]

In the previous problem we observed that the coefficients for a fourth degree binomial were found in the fifth row of Pascal’s Triangle. Here we have a 6\(^{th}\) degree binomial, so the coefficients will be found in the 7\(^{th}\) row of Pascal’s Triangle. Now we can fill in the blanks with the correct coefficients.

\[
x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6
\]

Examples

Example 1

Earlier, you were asked to expand the binomial \((x - 2)^5\).

To expand the binomial \((x - 2)^5\), you could use Pascal’s Triangle.

The degree of this expansion is 5, so the powers of \(x\) will begin with 5 and decrease by one each term until reaching 0 while the powers of \(y\), which in this case is \(-2\), will begin with zero and increase by one each term until reaching 5. We can write the variables in the expansion (leaving space for the coefficients) as follows:

\[
\phantom{x^5} + \phantom{x^4}y + \phantom{x^3}y^2 + \phantom{x^2}y^3 + \phantom{x}y^4 + \phantom{y}^5
\]

The coefficients for a fifth degree binomial can be found in the sixth row of Pascal’s Triangle. Now we can fill in the blanks with the correct coefficients, replacing \(y\) with \(-2\).
\[ 1 \cdot x^5 + 5x^4(-2) + 10x^3(-2)^2 + 10x^2(-2)^3 + 5x(-2)^4 + 1 \cdot (-2)^5 \]
\[ x^5 - 10x^4 + 40x^3 - 80x^2 + 80x - 32 \]

**Example 2**

Write out the elements in row 10 of Pascal’s Triangle.

We continued the triangle to find the \(9^{th}\) row earlier and determined it to be: 1 8 28 56 70 56 28 8 1

Subsequently, the \(10^{th}\) row is: 1 9 36 84 126 126 84 36 9 1

**Example 3**

Expand \((a + 4)^3\).

\[ a^3 + 3a^2(4) + 3a(4)^2 + (4)^3 \]
\[ a^3 + 12a^2 + 48a + 64 \]

**Example 4**

Write out the coefficients in the expansion of \((2x - 3)^4\).

\[ (2x)^4 + 4(2x)^3(-3) + 6(2x)^2(-3)^2 + 4(2x)(-3)^3 + (-3)^4 \]
\[ 16x^4 - 96x^3 + 216x^2 - 216x + 81 \]

**Review**

1. Write out the elements in row 7 of Pascal’s Triangle.
2. Write out the elements in row 13 of Pascal’s Triangle.

Use Pascal’s Triangle to expand the following binomials.

3. \((x - 6)^4\)
4. \((2x + 5)^6\)
5. \((3 - x)^7\)
6. \((x^2 - 2)^3\)
7. \((x+4)^5\)
8. \((2 - x^3)^4\)
9. \((a - b)^4\)
10. \((x + 1)^{10}\)

**Answers for Review Problems**

To see the Review answers, open this PDF file and look for section 12.6.

PDF file